

# A certifying algorithm for 3-colorability of $P_5$ -free graphs

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**Abstract.** We provide a certifying algorithm for the problem of deciding whether a  $P_5$ -free graph is 3-colorable by showing there are exactly six finite graphs that are  $P_5$ -free and not 3-colorable and minimal with respect to this property.

## 1 Introduction

An algorithm is *certifying* if it returns with each output a simple and easily verifiable certificate that the particular output is correct. For example, a certifying algorithm for the bipartite graph recognition would return either a 2-coloring of the input graph proving that it is bipartite, or an odd cycle proving it is not bipartite. A certifying algorithm for planarity would return a planar embedding or one of the two Kuratowski subgraphs. The notion of certifying algorithm [9] was developed when researchers noticed that a well known planarity testing program was incorrectly implemented. A certifying algorithm is a desirable tool to guard against incorrect implementation of a particular algorithm. In this paper, we give a certifying algorithm for the problem of deciding whether a  $P_5$ -free graph is 3-colorable. We will now discuss the background of this problem.

A class  $\mathcal{C}$  of graphs is called *hereditary* if for each graph  $G$  in  $\mathcal{C}$ , all induced subgraphs of  $G$  are also in  $\mathcal{C}$ . Every hereditary class of graphs can be described by its *forbidden induced subgraphs*, i.e. the unique set of minimal graphs which do not belong to the class. A comprehensive survey on coloring of graphs in hereditary classes can be found in [12]. An important line of research on colorability of graphs in hereditary classes deals with  $P_t$ -free graphs. The induced path on  $t$  vertices is called  $P_t$ , and a graph is called  *$P_t$ -free* if it does not contain  $P_t$  as an induced subgraph.

It is known that 4-COLORABILITY is NP-complete for  $P_9$ -free graphs [14] and 5-COLORABILITY is NP-complete for  $P_8$ -free graphs [10]. And most recently it was proved that 6-COLORABILITY is NP-complete for  $P_7$ -free [2]. On the other hand, the  $k$ -COLORABILITY problem can be solved in polynomial time for  $P_4$ -free graphs (since they are perfect). In [5] and [6], it is shown that  $k$ -COLORABILITY can be solved for the class of  $P_5$ -free graphs in polynomial time for every particular value of  $k$ . For  $t = 6, 7$ , the complexity of the problem is generally unknown, except for the case

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of 3-COLORABILITY of  $P_6$ -free graphs [13]. Known results on the  $k$ -COLORABILITY problem in  $P_t$ -free graphs are summarized in Table 1 ( $n$  is the number of vertices in the input graph,  $m$  the number of edges, and  $\alpha$  is matrix multiplication exponent known to satisfy  $2 \leq \alpha < 2.376$  [3]).

$k \setminus t$	3	4	5	6	7	8	9	10	11	12	...
3	$O(m)$	$O(m)$	$O(n^\alpha)$	$O(mn^\alpha)$	?	?	?	?	?	?	...
4	$O(m)$	$O(m)$	P	?	?	?	$NP_c$	$NP_c$	$NP_c$	$NP_c$	...
5	$O(m)$	$O(m)$	P	?	?	$NP_c$	$NP_c$	$NP_c$	$NP_c$	$NP_c$	...
6	$O(m)$	$O(m)$	P	?	$NP_c$	$NP_c$	$NP_c$	$NP_c$	$NP_c$	$NP_c$	...
7	$O(m)$	$O(m)$	P	?	$NP_c$	$NP_c$	$NP_c$	$NP_c$	$NP_c$	$NP_c$	...
...	...	...	...	...	...	...	...	...	...	...	...

**Table 1.** Known complexities for  $k$ -colorability of  $P_t$ -free graphs

In this paper, we study the coloring problem for the class of  $P_5$ -free graphs. This class has proved resistant with respect to other graph problems. For instance,  $P_5$ -free graphs is the unique minimal class defined by a single forbidden induced subgraph with unknown complexity of the MAXIMUM INDEPENDENT SET and MINIMUM INDEPENDENT DOMINATING SET problems. Many algorithmic problems are known to be NP-hard in the class of  $P_5$ -free graphs, for example DOMINATING SET [7] and CHROMATIC NUMBER [8]. In contrast to the NP-hardness of finding the chromatic number of a  $P_5$ -free graph, it is known [5] that  $k$ -COLORABILITY can be solved in this class in polynomial time for every particular value of  $k$ . This algorithm produces a  $k$ -coloring if one exists, but does not produce an easily verifiable certificate when such coloring does not exist. We are interested in finding a certificate for non- $k$ -colorability of  $P_5$ -free graphs. For this purpose, we start with  $k = 3$ .

Besides [5], there are several polynomial-time algorithms for 3-coloring a  $P_5$ -free graph ([6, 11, 14]) but none of them is a certifying algorithm. In this paper, we obtain a certifying algorithm for 3-coloring a  $P_5$ -free graphs by proving there are a finite number of minimally non-3-colorable  $P_5$ -free graphs and each of these graphs is finite.

**Theorem 1.1.** *A  $P_5$ -free graph is 3-colorable if and only if it does not contain any of the six graphs in Fig. 1 as a subgraph.*

It is an easy matter to verify the graphs in Fig. 1 are not 3-colorable, the rest of the paper involves proving the other direction of the theorem. In the last Section, we will discuss open problems arising from our work.

## 2 Definition and Background

Let  $k$  and  $t$  be positive integers. An MNkPt is a graph  $G$  that (i) is not  $k$ -colorable and is  $P_t$ -free and (ii) every proper subgraph of  $G$  is either  $k$ -colorable or has a  $P_t$ . We will be interested specifically in the case where  $k = 3$  and  $t = 5$ . We will use the following notations. Let  $G$  be a simple undirected graph. A set  $S$  of vertices of  $G$  is *dominating* if

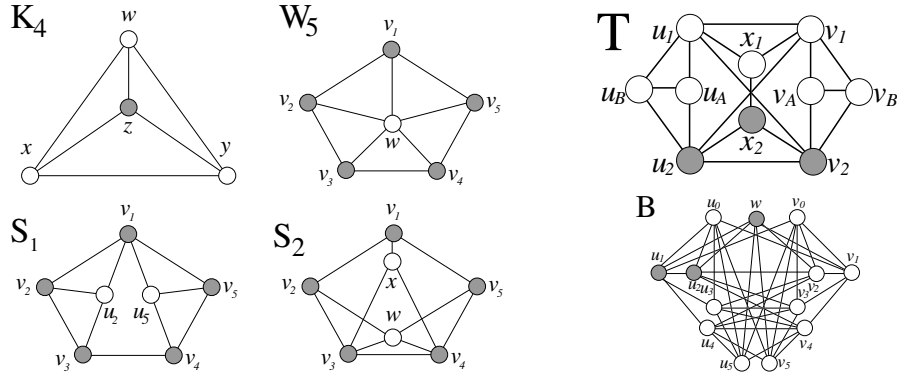


Fig. 1. All 6 MN3P5s

every vertex in  $G - S$  has a neighbor in  $S$ . A  $k$ -clique is a clique on  $k$  vertices.  $u \sim v$  will mean vertex  $u$  is adjacent to vertex  $v$ .  $u \not\sim v$  will mean vertex  $u$  is not adjacent to vertex  $v$ . For any vertex  $v$ ,  $N(v)$  denotes the set of vertices that are adjacent to  $v$ . We write  $G \cong H$  to mean  $G$  is isomorphic to  $H$ . The *clique number* of  $G$ , denoted by  $\omega(G)$ , is the number of vertices in a largest clique of  $G$ . The *chromatic number* of  $G$ , denoted by  $\chi(G)$ , is the smallest number of colors needed to color the vertices of  $G$ . A *hole* is an induced cycle with at least four vertices, and it is odd (or even) if it has odd (or even) length. An *anti-hole* is the complement of a hole. A  $k$ -hole ( $k$ -anti-hole) is a hole (anti-hole) on  $k$  vertices. A graph  $G$  is *perfect* if each induced subgraph  $H$  of  $G$  has  $\chi(H) = \omega(H)$ .

**Theorem 2.1 (The Strong Perfect Graph Theorem [4]).** *A graph is perfect if and only if it does not contain an odd hole or odd anti-hole as an induced subgraph.*

Let  $\mathcal{G} = \{K_4, W_5, S_1, S_2, T, B\}$  be the set of graphs in Fig. 1. We will denote these graphs in the following way.

- $P_5(v_1v_2v_3v_4v_5)$  means there is a  $P_5$  being  $v_1, v_2, v_3, v_4$  and  $v_5$ .
- $K_4(wxyz)$  means  $\{w, x, y, z\}$  form a  $K_4$ .
- $W_5(v_1v_2v_3v_4v_5, w)$  means  $v_1, v_2, v_3, v_4, v_5$  and  $w$  form a  $W_5$  where  $v_1v_2v_3v_4v_5$  form a 5-cycle and  $w$  is adjacent to every other vertex.
- $S_1(v_1v_2v_3v_4v_5, u_2, u_5)$  means  $v_1, v_2, v_3, v_4, v_5, u_2, u_5$  form an  $S_1$  where  $v_1$  is the only degree 4 vertex and  $N(v_1) = \{u_5, u_2, v_5, v_2\}$ . Also  $N(v_3) = \{v_4, v_2, u_2\}$  and  $N(v_4) = \{v_3, v_5, u_5\}$ , and  $v_1v_2v_3v_4v_5$  form a 5-cycle.
- $S_2(v_1v_2v_3v_4v_5, w, x)$  means  $v_1, v_2, v_3, v_4, v_5, w$  and  $x$  form an  $S_2$  where  $N(w) = \{v_2, v_3, v_4, v_5\}$ ,  $N(x) = \{v_1, v_3, v_4\}$  and  $v_1v_2v_3v_4v_5$  form a 5-cycle.
- $T(u_1u_Au_Bu_2, v_1v_Av_Bv_2, x_1, x_2)$  means a  $T$  graph is present as shown previously.
- $B(w, u_0u_1u_2u_3u_4u_5, v_0v_1v_2v_3v_4v_5)$  means a  $B$  graph is present as shown previously.

We will rely on the following result.

**Theorem 2.2 ([1]).** *Every connected  $P_5$ -free graph has a dominating clique or a dominating  $P_3$ .*

The following lemma is folklore.

**Lemma 2.1 (The neighborhood lemma).** *Let  $G$  be a minimally non  $k$ -colorable graph. If  $u$  and  $v$  are two non-adjacent vertices in  $G$ , then  $N(u) \not\subseteq N(v)$ .*

*Proof.* Assume  $N(u) \subseteq N(v)$ . Then the graph  $G - v$  admits a  $k$ -coloring. By giving  $u$  the color of  $v$ , we see that  $G$  is  $k$ -colorable, a contradiction.  $\square$

The neighborhood lemma is used predominantly throughout this paper. Writing  $N(\mathbf{v}, \mathbf{w}) \rightarrow \mathbf{u}$  will denote the fact that  $N(v) \not\subseteq N(w)$  by the neighborhood lemma so there exists a vertex  $u$  where  $u \sim v$ , but  $u \not\sim w$ .

The following fact is well-known and easy to establish.

**Fact 2.1.** *In a minimally non  $k$ -colorable graph every vertex has degree at least  $k$ .*  $\square$

### 3 Intermediate Results

In this section, we establish a number of intermediate results needed for proving the main theorem.

**Lemma 3.1.** *Let  $G$  be an MN3P5 graph with a 5-hole  $C = \{v_1, v_2, v_3, v_4, v_5\}$  and a vertex  $w$  adjacent to at least 4 vertices of  $C$ . Then  $G \in \mathfrak{G}$ .*

*Proof.* If  $w$  is adjacent to all five vertices of  $C$ , then  $G$  clearly is isomorphic to  $W_5$ . Now, assume  $N(w) \cap \{v_1, v_2, v_3, v_4, v_5\} = \{v_2, v_3, v_4, v_5\}$ .

We have  $N(\mathbf{v}_1, \mathbf{w}) \rightarrow \mathbf{x}$ .

Assume for the moment that  $x \not\sim v_3, v_4$ . We have

$x \sim v_5$ , otherwise, we have  $P_5(xv_1v_5v_4v_3)$ .

$x \sim v_2$ , otherwise, we have  $P_5(xv_1v_2v_3v_4)$ .

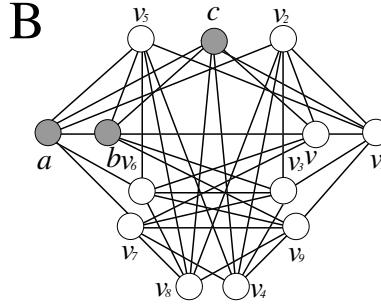
But then  $G$  contains  $S_1(v_1v_2v_3v_4v_5, x, w)$ . This means  $x \sim v_3$  or  $x \sim v_4$ . By symmetry, we may assume  $x \sim v_3$ . We have  $x \sim v_2$  or  $x \sim v_4$ , otherwise,  $G$  contains  $P_5(xv_1v_2wv_4)$ . If  $x \sim v_2$  then  $G$  properly contains  $S_1(v_1v_2v_3v_4v_5, x, w)$ , a contradiction. This means  $x \sim v_4$ ; so  $G$  contains  $S_2(v_1v_2v_3v_4v_5, w, x)$  and  $G \cong S_2$ .  $\square$

**Theorem 3.1.** *Every MN3P5 graph different from  $K_4$  contains a 5-hole.*

*Proof.* Let  $G$  be an MN3P5 graph different from a  $K_4$ . We have  $\omega(G) \leq 3$  and  $\chi(G) \geq 4$ . Thus,  $G$  is not perfect. By Theorem 2.1,  $G$  contains an odd hole or an odd anti-hole  $H$ .  $H$  cannot be a hole of size 7 or greater because  $G$  is  $P_5$ -free. We may assume  $H$  is an anti-hole of length at least seven, for otherwise we are done (observe that the hole on five vertices is self-complementary). Let  $v_1, v_2, v_3, v_4, v_5, v_6, v_7$  be the cyclic order of the hole in the complement of  $G$ . Then  $G$  properly contains  $S_1(v_4v_6v_3v_5v_2, v_1, v_7)$ , a contradiction.  $\square$

**Lemma 3.2.** *Let  $G$  be an MN3P5 graph that has a dominating clique  $\{a, b, c\}$ . Also assume that there is a vertex  $v \notin \{a, b, c\}$  adjacent to two vertices from  $\{a, b, c\}$ . Then  $G \in \mathfrak{G}$ .*

*Proof.* The proof is by contradiction. Suppose that  $G \notin \mathfrak{G}$ . We may assume  $v$  is adjacent to  $b$  and  $c$ . We have  $v \approx a$ , otherwise,  $G$  contains  $K_4(abcv)$ . Through repeated applications of the Neighborhood Lemma, we will eventually add nine vertices to  $G$  to arrive at a contradiction. In the end, we will obtain the graph B (see Fig. 2 for the order in which vertices are added). Each time we add a vertex we will consider its adjacency to the other vertices of the graph. In every case, the adjacency can be completely determined at each step.



**Fig. 2.** The graph B obtained in the proof of Lemma 3.2

$\mathbf{N}(v, a) \rightarrow v_1$ .

- $v_1 \sim c$ : since  $\{a, b, c\}$  is dominating,  $v_1$  is adjacent to either  $b$  or  $c$ . Without loss of generality, assume  $v_1 \sim c$ .
- $v_1 \approx b$ : otherwise,  $G$  contains  $K_4(bc v v_1)$ .

$\mathbf{N}(v_1, b) \rightarrow v_2$ .

- $v_2 \sim a$ : assume  $v_2 \approx a$ . We have  $v_2 \sim v$ , otherwise,  $G$  contains  $P_5(v_2 v_1 v b a)$ . Also,  $v_2 \sim c$  since  $\{a, b, c\}$  is a dominating set. But then,  $G$  contains  $K_4(v_1 v_2 v c)$ .
- $v_2 \approx c$ : otherwise,  $G$  contains  $W_5(ab v v_1 v_2, c)$ .
- $v_2 \sim v$ : otherwise,  $c$  has four neighbors in the 5-hole  $v_2 a b v v_1$  contradicting Lemma 3.1.

$\mathbf{N}(v_2, c) \rightarrow v_3$ .

- $v_3 \sim b$ : assume  $v_3 \approx b$ . We have  $v_3 \sim a$  since  $\{a, b, c\}$  is a dominating set. We have  $v_3 \approx v_1$ , otherwise,  $G$  contains  $S_1(v b a v_3 v_2, c, v_1)$ . But then  $G$  contains  $P_5(v_3 v_2 v_1 c b)$ .
- $v_3 \approx v$ : otherwise,  $G$  contains  $W_5(bc v_1 v_2 v_3, v)$ .
- $v_3 \sim v_1$ : otherwise,  $v$  has four neighbors in the 5-hole  $v_3 b c v_1 v_2$  contradicting Lemma 3.1.
- $v_3 \approx a$ : otherwise  $G$  contains  $S_1(v_3 a c v v_1, b, v_2)$ .

$\mathbf{N}(\mathbf{v}_3, \mathbf{v}) \rightarrow \mathbf{v}_4$ .

- $v_4 \sim c$ : assume  $v_4 \approx c$ . Then we have  $v_4 \sim v_2$ , for otherwise  $G$  contains  $P_5(v_4v_3v_2vc)$ ;  $v_4 \approx v_1$ , for otherwise  $G$  contains  $K_4(v_1v_2v_3v_4)$ ;  $v_4 \approx b$ , for otherwise  $G$  contains  $S_1(v_1v_2v_4bc, v_3, v)$ ;  $v_4 \sim a$  because  $\{a, b, c\}$  is dominating. But then  $G$  contains  $P_5(v_4abvv_1)$ .
- $v_4 \approx v_1$ : for otherwise  $G$  contains  $W_5(v_4v_3v_2vc, v_1)$ .
- $v_4 \approx b$ : for otherwise,  $G$  contains  $S_1(vv_1v_3v_4b, v_2, c)$ .
- $v_4 \approx a$ : for otherwise,  $G$  contains  $P_5(v_4abvv_1)$ .
- $v_4 \sim v_2$ : for otherwise the vertex  $v_1$  has exactly four neighbors in the 5-hole  $v_4v_3v_2vc$  contradicting Lemma 3.1.

$\mathbf{N}(\mathbf{a}, \mathbf{v}) \rightarrow \mathbf{v}_5$ .

- $v_5 \approx v_3$ : Assume  $v_5 \sim v_3$ . Then we have  $v_5 \sim v_1$ , for otherwise  $G$  contains  $P_5(av_5v_3v_1v)$ ;  $v_5 \approx v_2$ , for otherwise  $G$  contains  $K_4(v_1v_2v_3v_5)$ ;  $v_5 \sim c$ , for otherwise  $G$  contains  $P_5(v_5v_3v_2vc)$ . But now  $G$  contains  $W_5(v_5cvv_2v_3, v_1)$ .
- $v_5 \sim b$ : assume  $v_5 \approx b$ . Then we have  $v_5 \sim v_1$ , for otherwise  $G$  contains  $P_5(v_5abvv_1)$ . But then  $c$  has four neighbors in the 5-hole  $v_5abvv_1$  contradicting Lemma 3.1.
- $v_5 \approx c$ : for otherwise  $G$  contains  $K_4(abcv_5)$ .
- $v_5 \sim v_1$ : for otherwise  $G$  contains  $P_5(v_5acv_1v_3)$ .
- $v_5 \sim v_4$ : for otherwise  $G$  contains  $P_5(v_3v_4cav_5)$ .
- $v_5 \approx v_2$ : for otherwise  $G$  contains  $S_1(cvv_2v_5a, v_1, b)$ .

$\mathbf{N}(\mathbf{v}_5, \mathbf{c}) \rightarrow \mathbf{v}_6$ .

- $v_6 \sim v$ : assume  $v_6 \approx v$ . We have  $v_6 \sim a$ , for otherwise  $G$  contains  $P_5(v_6v_5acv)$ ;  $v_6 \approx b$ , for otherwise  $G$  contains  $K_4(abv_5v_6)$ ;  $v_6 \sim v_1$ , for otherwise,  $G$  contains  $P_5(v_6abvv_1)$ . But  $c$  has four neighbors in the 5-hole  $v_6abvv_1$  contradicting Lemma 3.1.
- $v_6 \approx b$ : for otherwise  $G$  contains  $W_5(v_5v_6vca, b)$ .
- $v_6 \approx v_2$ : for otherwise  $G$  contains  $P_5(v_2v_6v_5bc)$ .
- $v_6 \sim v_3$ : for otherwise  $G$  contains  $P_5(v_5v_6vv_2v_3)$ .
- $v_6 \sim a$ : for otherwise  $G$  contains  $P_5(v_3v_6v_5ac)$ .
- $v_6 \approx v_1$ : for otherwise  $G$  contains  $S_1(v_6abcv, v_5, v_1)$ .
- $v_6 \approx v_4$ : for otherwise  $G$  contains  $T(v_6av_5b, v_3v_2v_1v, v_4, c)$ .

$\mathbf{N}(\mathbf{v}_4, \mathbf{v}_1) \rightarrow \mathbf{v}_7$ .

- $v_7 \sim v$ : assume  $v_7 \approx v$ . Then we have  $v_7 \sim v_3$ , for otherwise  $G$  contains  $P_5(v_7v_4v_3v_1v)$ ;  $v_7 \approx v_2$ , for otherwise  $G$  contains  $K_4(v_2v_3v_4v_7)$ ;  $v_7 \sim c$ , for otherwise  $G$  contains  $P_5(v_7v_3v_2vc)$ . Now,  $G$  contains  $S_1(v_2vcv_7v_3, v_1, v_4)$ .
- $v_7 \approx v_2$ : for otherwise  $G$  contains  $W_5(vv_1v_3v_4v_7, v_2)$ .
- $v_7 \approx v_6$ : for otherwise  $G$  contains  $P_5(v_6v_7v_4v_2v_1)$ .
- $v_7 \sim a$ : for otherwise  $G$  contains  $P_5(v_4v_7vv_6a)$ .
- $v_7 \sim v_3$ : for otherwise  $G$  contains  $P_5(av_7vv_1v_3)$ .
- $v_7 \approx c$ : for otherwise  $G$  contains  $S_1(v_3v_4cvv_1, v_7, v_2)$ .
- $v_7 \approx b$ : for otherwise  $G$  contains  $P_5(v_7bcv_1v_2)$ .
- $v_7 \approx v_5$ : for otherwise  $G$  contains  $T(av_7v_5v_4, cvv_1v_2, b, v_3)$ .

$\mathbf{N}(\mathbf{v}_6, \mathbf{b}) \rightarrow \mathbf{v}_8$ .

- $v_8 \sim c$ : assume  $v_8 \approx c$ . Then we have  $v_8 \sim a$  because  $\{a, b, c\}$  is a dominating set;  $v_8 \sim v_5$ , for otherwise  $G$  contains  $P_5(v_8v_6v_5bc)$ . But now,  $G$  contains  $K_4(av_5v_6v_8)$ .
- $v_8 \approx a$ : for otherwise  $G$  contains  $W_5(v_8v_6v_5bc, a)$ .
- $v_8 \approx v_1$ : for otherwise  $G$  contains  $P_5(bav_6v_8v_1)$ .
- $v_8 \sim v_2$ : for otherwise  $G$  contains  $P_5(v_2v_1cv_8v_6)$ .
- $v_8 \sim v_5$ : for otherwise  $G$  contains  $P_5(v_8v_2v_1v_5b)$ .
- $v_8 \approx v_4$ : for otherwise  $G$  contains  $P_5(v_4v_8v_6ab)$ .
- $v_8 \approx v$ : for otherwise  $G$  contains  $S_1(bcv_8v_6a, v, v_5)$ .
- $v_8 \approx v_3$ : for otherwise  $G$  contains  $T(v_6av_5b, v_3v_2v_1v, v_8, c)$ .
- $v_8 \sim v_7$ : for otherwise  $G$  contains  $P_5(v_8v_5bv_3v_7)$ .

$\mathbf{N}(\mathbf{v}_8, \mathbf{a}) \rightarrow \mathbf{v}_9$ .

- $v_9 \sim b$ : assume  $v_9 \approx b$ . We have  $v_9 \sim v_2$ , for otherwise  $G$  contains  $P_5(v_9v_8v_2ab)$ ;  $v_9 \sim v_6$ , for otherwise  $G$  contains  $P_5(v_9v_8v_6ab)$ . This means  $G$  contains  $T(v_6v_8v_9v_2, abcv, v_5, v_1)$ .
- $v_9 \sim v_1$ : assume  $v_9 \approx v_1$ . We have  $v_9 \sim v_2$ , for otherwise  $G$  contains  $P_5(v_9bav_2v_1)$ . This means  $G$  contains  $T(v_2vv_1c, v_8v_6v_5a, v_9, b)$ .
- $v_9 \sim v_6$ : for otherwise  $G$  contains  $P_5(v_1v_9bav_6)$ .
- $v_9 \sim v_7$ : for otherwise  $G$  contains  $P_5(v_1v_9bav_7)$ .
- $v_9 \sim v_4$ : assume  $v_9 \approx v_4$ . Then we have  $v_9 \sim v_2$ , for otherwise  $G$  contains  $P_5(v_9bav_2v_4)$ . This means  $G$  contains  $T(v_6av_5b, v_9v_2v_1v, v_8, c)$ .

But this means  $G$  contains  $B(c, v_5abv_6v_7v_8, v_2v_1vv_3v_9v_4)$ , a contradiction.  $\square$

**Lemma 3.3.** *Let  $G$  be an MN3P5 with a dominating clique  $\{a, b, c\}$ . Let  $A = N(a) - \{b, c\}$ ,  $B = N(b) - \{a, c\}$  and  $C = N(c) - \{a, b\}$ . Suppose  $A$ ,  $B$  and  $C$  are pairwise disjoint. Then  $G \in \mathfrak{G}$ .*

*Proof.* Some observations are necessary for this proof.

**Observation 3.1.** *Let  $X$  and  $Y$  be two distinct elements of  $\{A, B, C\}$ . Let  $X'$  be a component in  $X$  with at least two vertices, and  $y$  be a vertex in  $Y$ . Then either  $y$  is adjacent to all vertices of  $X'$  or to no vertex of  $X'$ .*

*Proof.* Suppose the Observation is false. Then there are adjacent vertices  $v_1, v_2 \in X$  such that  $y$  is adjacent to exactly one of  $v_1, v_2$ . Without loss of generality, we may assume  $X = A$  and  $Y = B$ . Now,  $\{c, b, y, v_2, v_1\}$  induces a  $P_5$ , a contradiction.  $\square$

**Observation 3.2.** *Every component in  $A$ ,  $B$  or  $C$  is a single edge or one vertex.*

*Proof.* Assume that one of  $A$ ,  $B$  or  $C$  contains a vertex of degree 2. Without loss of generality, assume there is such a vertex  $a_0 \in A$  that is adjacent to two other distinct vertices  $a_1$  and  $a_x$  in  $A$ . Now we have  $a_1 \approx a_x$ , for otherwise  $G$  contains  $K_4(a_1a_xa_0a)$ . The Neighborhood Lemma implies  $\mathbf{N}(\mathbf{a}_1, \mathbf{a}_x) \rightarrow \mathbf{a}_2$  and  $\mathbf{N}(\mathbf{a}_x, \mathbf{a}_1) \rightarrow \mathbf{a}_y$ . Observation 3.1 implies  $a_2, a_y \in A$ . We have  $a_y \approx a_0$ , for otherwise  $G$  contains  $K_4(aa_0a_xa_y)$ ;  $a_2 \approx a_0$ , for otherwise  $G$  contains  $K_4(aa_0a_1a_2)$ ;  $a_y \sim a_2$ , for otherwise  $G$  contains  $P_5(a_ya_xa_0a_1a_2)$ . Then  $G$  contains  $W_5(a_ya_xa_0a_1a_2, a)$ , a contradiction.  $\square$

We continue the proof of the Lemma. Assume  $G \notin \mathfrak{G}$ . Consider the case that two of  $A, B$  or  $C$  contain an edge. Without loss of generality, assume  $A$  contains an edge  $a_1a_2$  and  $B$  contains an edge  $b_1b_2$ . If a vertex in  $\{b_1, b_2\}$  is adjacent to a vertex in  $\{a_1, a_2\}$  then by Observation 3.1,  $G$  contains  $K_4(a_1a_2b_1b_2)$ , a contradiction. Suppose some vertex  $c_0 \in C$  is adjacent to a vertex in  $\{a_1, a_2, b_1, b_2\}$ . We may assume  $c_0 \sim a_1$ . By Observation 3.1, we have  $c_0 \sim a_2$ . If  $c_0 \approx b_i$  ( $i = 1, 2$ ) then  $G$  contains  $P_5(b_ibc_0a_1)$ . So,  $c_0$  is adjacent to all vertices of  $\{a_1, a_2, b_1, b_2\}$ . But now,  $G$  contains  $S_1(c_0a_1abb_1, a_2, b_2)$ . So, no vertex in  $C$  is adjacent to a vertex in  $\{a_1, a_2, b_1, b_2\}$ . By Fact 2.1 and Observation 3.2, there exists a vertex  $a_3 \in A$  with  $b_1, b_2 \sim a_3$  and a vertex  $b_3 \in B$  with  $a_1, a_2 \sim b_3$ . Also by Fact 2.1,  $C$  contains a vertex  $c_0$ . We have  $a_3 \sim c_0$ , for otherwise  $G$  contains  $P_5(a_3b_1bcc_0)$ ;  $b_3 \sim c_0$ , for otherwise  $G$  contains  $P_5(b_3a_1acc_0)$ ;  $a_3 \sim b_3$ , for otherwise  $G$  contains  $P_5(b_1a_3c_0b_3a_1)$ . But now  $G$  contains  $T(aa_1a_2b_3, bb_1b_2a_3, c, c_0)$  which is a contradiction. So, at most one of  $A, B, C$  contains an edge.

If all of  $A, B, C$  is a stable set, then  $G$  is obviously 3-colorable. We may assume  $B, C$  are stable sets, and  $A$  contains an edge. Now there must be one vertex  $b_0 \in B$  with  $N(b_0)$  contains two adjacent vertices in  $A$ . Otherwise,  $G$  admits a 3-coloring  $f$  as follows. The vertices of  $C$  are colored with color 3. Now, for each edge in  $A$ , its endpoints are arbitrarily colored with colors 1, 2. The remaining vertices of  $A$  are colored with color 1. The vertices of  $B$  are colored with color 2 (no vertex of  $B$  is adjacent to an endpoint of a edge of  $A$  by Observation 3.1), and let  $f(a) = 3, f(b) = 1, f(c) = 2$ . Thus,  $f$  is a 3-coloring which is a contradiction. Therefore, there is a vertex  $b_1 \in B$  adjacent to both endpoints in some edge  $a_{b_1}a_{b_2}$  in  $A$ . By a similar argument, there is a vertex  $c_1 \in C$  adjacent to both endpoints in some edge  $a_{c_1}a_{c_2}$ .

Suppose that  $a_{b_1}a_{b_2}$  and  $a_{c_1}a_{c_2}$  are the same edge. For simplicity, write  $a_1a_2 = a_{b_1}a_{b_2} = a_{c_1}a_{c_2}$ . We have  $b_1 \approx c_1$ , for otherwise  $G$  contains  $K_4(a_1a_2b_1c_1)$ .

- $N(\mathbf{b}_1, \mathbf{a}) \rightarrow \mathbf{c}_2$ . We have  $c_2 \in C$  by the fact that  $B$  is an independent set.
- $N(\mathbf{c}_1, \mathbf{a}) \rightarrow \mathbf{b}_2$ . We have  $b_2 \in B$  by the fact that  $C$  is an independent set.
- $b_2, c_2 \approx a_1, a_2$ . Otherwise, suppose  $b_2 \sim a_1$ . Then by Observation 3.1, we have  $b_2 \sim a_2$  so  $G$  contains  $K_4(a_1a_2b_2c_1)$ .
- $b_2 \sim c_2$ . Otherwise,  $G$  contains  $P_5(c_1b_2bb_1c_2)$ .

Now,  $G$  contains  $P_5(b_2c_2caa_1)$ . Thus,  $a_{b_1}a_{b_2}$  and  $a_{c_1}a_{c_2}$  are distinct edges. We have  $b_1 \approx a_{c_1}, a_{c_2}$  and  $c_1 \approx a_{b_1}, a_{b_2}$ , for otherwise we are done by the previous case. We have  $b_1 \sim c_1$ , for otherwise  $G$  contains  $P_5(b_1a_{b_1}aa_{c_1}c_1)$ . But now  $G$  contains  $S_1(ab_1a_{b_1}b_1c_1a_{c_1}, a_{b_2}, a_{c_2})$ , a contradiction.  $\square$

**Lemma 3.4.** *Let  $G$  be an MN3P5 with a dominating clique  $\{a, b, c\}$ . Then  $G \in \mathfrak{G}$ .*

*Proof.* If there is a vertex other than  $a, b$  and  $c$  adjacent to at least two of  $a, b$  or  $c$  then by Lemma 3.2,  $G \in \mathfrak{G}$ . Otherwise, the conclusion follows from Lemma 3.3.  $\square$

**Lemma 3.5.** *Let  $G$  be an MN3P5 with a dominating clique  $\{a, b\}$  of size 2. Then  $G \in \mathfrak{G}$ .*

*Proof.* Assume  $G \notin \mathfrak{G}$ . We may assume  $G$  contains no dominating 3-clique, for otherwise we are done by Lemma 3.4. It follows that no vertex  $v$  is adjacent to both  $a, b$ .



By Theorem 3.1, there is 5-hole  $C = v_1v_2v_3v_4v_5$  in  $G$  because  $G \neq K_4$ . Clearly  $C$  cannot contain both  $a$  and  $b$ . WLOG, assume that  $|N(a) \cap C| \geq |N(b) \cap C|$ . If  $b \notin C$  then since  $\{a, b\}$  is a dominating clique of  $G$  we have  $|N(a) \cap C| \geq 3$ . If  $b \in C$ , then  $a$  must be adjacent to the 2 vertices in  $C$  not adjacent to  $b$ . Thus, since  $a \sim b$  we also have  $|N(a) \cap C| \geq 3$ . The case when  $|N(a) \cap C| \geq 4$  is handled by Lemma 3.1, so WLOG we may assume either  $N(a) \cap C = \{v_1, v_2, v_3\}$  or  $N(a) \cap C = \{v_1, v_3, v_4\}$ .

Suppose  $N(a) \cap C = \{v_1, v_2, v_3\}$ . Since  $\{a, b\}$  is a dominating clique, we have  $b \notin C$  and  $b \sim v_4, v_5$ . Since no vertex is adjacent to both  $a$  and  $b$ ,  $G$  contains  $P_5(bv_5v_1v_2v_3)$ , a contradiction. Now, we may assume  $N(a) \cap C = \{v_1, v_3, v_4\}$ . There exists a vertex  $x$  with  $x \sim a, v_3, v_4$ , for otherwise  $\{a, v_3, v_4\}$  is dominating 3-clique. If  $x \sim v_5$ , then  $x \sim v_2$ , for otherwise  $G$  contains  $P_5(xv_5v_4v_3v_2)$ ; but now  $G$  contains  $P_5(v_2xv_5v_4a)$ . Thus, we have  $x \sim v_5$  and by symmetry  $x \sim v_2$ . Since  $\{a, b\}$  is a dominating clique, we have  $x \sim b$ , and  $b \sim v_2, v_5$ . Recall that no vertex is adjacent to both  $a, b$ . Now,  $G$  contains  $P_5(xbv_5v_4v_3)$  which is a contradiction.  $\square$

**Theorem 3.2.** *If  $G$  is an MN3P5 with a dominating clique then  $G \in \mathfrak{G}$ .*

*Proof.* If  $G$  has a dominating clique of size one or two, then it has a dominating clique of size 2 since  $G$  contains no isolated vertices. By Lemma 3.5,  $G \in \mathfrak{G}$ . If  $G$  has a dominating clique of size 3, then Lemma 3.4 implies  $G \in \mathfrak{G}$ . If  $G$  has a dominating clique of size 4 or more, then  $G$  contains a  $K_4$  so  $G = K_4 \in \mathfrak{G}$  by minimality.  $\square$

**Lemma 3.6.** *Let  $G$  be an MN3P5 with a dominating 5-hole. Then  $G$  has a dominating  $K_3$  or  $G \in \mathfrak{G}$ .*

*Proof.* Let  $C = v_1v_2v_3v_4v_5$  be an induced 5-hole of  $G$ . Assume  $G$  does not have a dominating clique. Let  $X_i$  be the set of vertices adjacent to  $v_{i-1}$  and  $v_{i+1}$  and not adjacent to  $v_{i+2}$  and  $v_{i+3}$  with the subscript taken modulo 5 (i.e.,  $v_0 = v_5$ ), for  $i = 1, 2, 3, 4, 5$ . We now prove every vertex of  $G$  belongs to exactly one  $X_i$ .

Consider a vertex  $w \notin C$ . By Lemma 3.1, we have  $1 \leq |N(w) \cap C| \leq 3$ . If  $w$  has one neighbor in  $C$ , then  $G$  obviously contains a  $P_5$ . Suppose  $w$  has two neighbors  $a, b$  in  $C$ . If  $a \sim b$ , then  $G$  obviously contains a  $P_5$ . Otherwise,  $a$  and  $b$  have distance two on  $C$  and so  $w$  belongs to some  $X_i$ . We may now assume  $w$  has three neighbors on  $C$ . If these three neighbors are consecutive on  $C$ , then  $w$  belongs to some  $X_i$ . Now, we may assume  $w \sim v_1, v_3, v_4$ . There is a vertex  $x$  with  $x \sim w, v_4, v_3$ , for otherwise  $\{w, v_4, v_3\}$  is a dominating clique. Vertex  $x$  must have a neighbor in  $\{v_1, v_2, v_5\}$  because  $C$  is a dominating set. If  $x \sim v_5$ , then  $x \sim v_2$ , for otherwise  $G$  contains  $P_5(xv_5v_4v_3v_2)$ ; but now  $G$  contains  $P_5(v_2xv_5v_4w)$ . Thus, we have  $x \sim v_5$  and by symmetry  $x \sim v_2$ . Now, we have  $x \sim v_1$ , and  $G$  contains  $P_5(xv_1v_5v_4v_3)$ . Thus,  $X_1, X_2, X_3, X_4, X_5$  is a partition of  $V(G)$ .

If there are nonadjacent vertices  $x_1, x_2$  with  $x_1 \in X_1, x_2 \in X_2$ , then  $G$  contains  $P_5(x_1v_5v_4v_3x_2)$ . Thus, there are all possible edges between  $X_i$  and  $X_{i+1}$  for all  $i$ . If every  $X_i$  is a stable set, then  $G$  is obviously 3-colorable, a contradiction. So we may assume WLOG  $X_5$  contains an edge  $ab$ . Then  $X_1$  is a stable set, for otherwise  $G$  contains a  $K_4$  with one edge in  $X_1$  and one edge in  $X_5$ . Similarly,  $X_4$  is a stable set. If  $X_2$  contains an edge  $cd$ , then  $G$  contains  $S_1(v_1cv_3v_4a, d, b)$ . If  $X_3$  contains an edge  $fg$ , then  $G$  contains  $S_1(v_4fv_2v_1a, g, b)$ . Thus,  $X_i$  is a stable set for  $i = 1, 2, 3, 4$ . Consider

the subgraph  $H$  of  $G$  induced by  $X_5$ . If  $H$  contains an odd cycle  $D$ , then  $D \cup \{v_1\}$  is a  $K_4$  or  $W_5$ , or  $D$  contains a  $P_5$ . Thus  $H$  is bipartite. By coloring  $X_5$  with colors 2,3,  $X_1 \cup X_4$  with color 1,  $X_2$  with color 2,  $X_3$  with color 3, we see that  $G$  is 3-colorable, a contradiction.  $\square$

## 4 Proof of Theorem 1.1

We can now prove the main theorem.

It is a routine matter to verify the “only if” part. We only need prove the “if” part. Suppose  $G$  does not contain any of the graphs in Fig. 1 but is not 3-colorable. Then  $G$  contains an induced subgraph that is minimally not 3-colorable. It follows that we may assume  $G$  is a connected MN3P5 graphs. By Theorem 2.2,  $G$  contains a dominating clique or  $P_3$ . If  $G$  contains a dominating clique, then we are done by Theorem 3.2. So, we may assume  $G$  contains no dominating clique and thus contains a dominating  $P_3$  with vertices  $v_1, v_2, v_3$  and edges  $v_1v_2, v_2v_3$ . There is a vertex  $v_4$  with  $v_4 \sim v_3$  and  $v_4 \not\sim v_1, v_2$  since  $v_1v_2$  is not a dominating edge. Similarly, there is a vertex  $v_5$  with  $v_5 \sim v_1$  and  $v_5 \not\sim v_2, v_3$ . We have  $v_5 \sim v_4$ , for otherwise  $G$  contains a  $P_5$ . Thus,  $v_1v_2v_3v_4v_5$  is a dominating 5-hole of  $G$ , and we are done by Lemma 3.6.  $\square$

## 5 Conclusion and Open Problems

In this paper, we provide a certifying algorithm for the problem of 3-coloring a  $P_5$ -graph by showing there are exactly six finite minimally non-3-colorable graphs. Previously known algorithms ([6, 11, 14]) provide a yes-certificate by constructing a 3-coloring if one exists. Our algorithm provides a no-certificate by finding one of the six graphs of Fig. 1. Since these graphs are finite, our algorithm runs in polynomial time. We do not know if there is a fast algorithm running in, say,  $O(n^4)$  to test if a graph contains one of the six graphs of Fig. 1 as a subgraph. We leave this as an open problem.

In [5, 6], it is shown for every fixed  $k$ , determining if a  $P_5$ -free graph is  $k$ -colorable is polynomial-time solvable. It is tempting to speculate that these two algorithms work because for every fixed  $k$ , there is a function  $f(k)$  such that every minimally non- $k$ -colorable  $P_5$ -free graph has at most  $f(k)$  vertices. The result of this paper can be viewed as a first step in this direction.

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