Constructions of k-critical P_5-free graphs

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Abstract

With respect to a class $\mathcal{C}$ of graphs, a graph $G \in \mathcal{C}$ is said to be *k-critical* if every proper subgraph of G belonging to $\mathcal{C}$ is $k-1$ colorable. We construct an infinite set of k-critical P_5-free graphs for every $k \geq 5$. We also prove that there are exactly eight 5-critical $\{P_5, C_5\}$-free graphs.

1 Introduction

Let P_t denote the chordless path on t vertices. The CHROMATIC NUMBER problem for P_5-free graphs is known to be NP-hard [6]. However for fixed k, the k-colorability question for P_5-free graphs can be answered in polynomial time [4, 5]. More generally, the k-colorability question for P_t-free graphs has been well studied [1, 4, 5, 8, 7, 9, 10]. The polynomial time algorithms for answering the k-colorability question for P_5-free graphs will return a valid k-coloring if one exists, but otherwise do not provide a *no-certificate* – or a minimal obstruction that makes the graph non k-colorable. This motivates the following research question: *Is there a forbidden subgraph characterization of k-colorable P_5-free graphs for fixed k?* When $k = 3$, the answer is "yes" and the 6 forbidden subgraphs are shown in Figure 1 [2]. This result is extended in [8] where they outline six additional forbidden *induced* subgraphs for 3-colorable P_5-free graphs. The six extra graphs are obtained by adding edges to the graphs in Figure 1, so that the graphs remain 4-colorable and P_5-free. In this paper, we investigate this question for $k > 3$.

Suppose a graph G has chromatic number k (i.e., G is k-colorable, but not $(k-1)$-colorable). Then G is said to be *k-vertex-critical* if removing any vertex from G results in a graph that is

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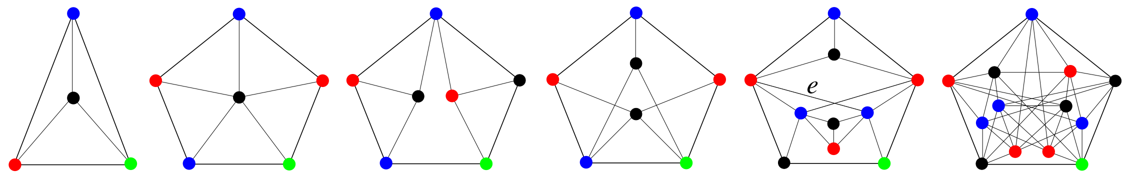

Figure 1: A P_5-free graph is 3-colorable if it does not contain any of the above graphs as a subgraph.

$(k-1)$-colorable. Observe that each of the six P_5-free graphs in Figure 1 are 4-vertex-critical. However, so are the additional 6 P_5-free graphs listed in [8]. Thus, for our question, the definition of "k-vertex-critical" is not strict enough. Traditionally, a graph G is said to be "k-critical" if every proper subgraph of G is $(k-1)$-colorable. This definition, however, is still insufficient for our purpose since the removal of the edge e from the 5-th graph listed in Figure 1 results in a graph that is not 3-colorable. The resulting graph, however, is no longer P_5-free. Therefore, we introduce a new definition with respect to a specific class of graphs.

Definition 1.1 *With respect to a class $\mathcal{C}$ of graphs, a graph $G \in \mathcal{C}$ is* k-critical *if every proper subgraph of G belonging to $\mathcal{C}$ is $k-1$ colorable.*

For the remainder of the paper, "k-critical" means "k-critical with respect to the considered class $\mathcal{C}$". Using Definition 1.1, the set of all 4-critical P_5-free graphs are precisely those listed in Figure 1. Note that this definition implies that all k-critical graphs are also k-vertex-critical. We now restate our original research question: *Are there a finite number of k-critical P_5-free graphs for fixed k?* While considering this question, the following results are obtained (where C_t denotes the chordless cycle on t vertices):

1. We prove that given a class $\mathcal{C}$ of graphs: if an infinite number are k-vertex-critical then an infinite number are k-critical.

2. We construct an infinite set of k-vertex-critical P_5-free graphs, for each $k \geq 5$.

3. We construct an infinite set of 5-critical P_5-free graphs.

4. We prove that there are exactly eight 5-critical $\{P_5, C_5\}$-free graphs.

Together, the first two results answer our modified research question. The final result was motivated by the observation that the graphs in our infinite set of 5-vertex-critical P_5-free graphs all contained a C_5 (for sufficiently large graphs). We note that it is NP-hard to k-color a $\{P_5, C_5\}$-free graph when k is part of the input [6].

In Section 2 we prove the first 3 results. In Section 3 we present an algorithm that is used to prove our final result. We conclude the paper in Section 4 with a number of interesting open problems.

2 P_5-free graphs

We will prove that the following construction produces an infinite set of 5-vertex-critical P_5-free graphs. $N(v)$ denotes the neighbourhood of vertex v.

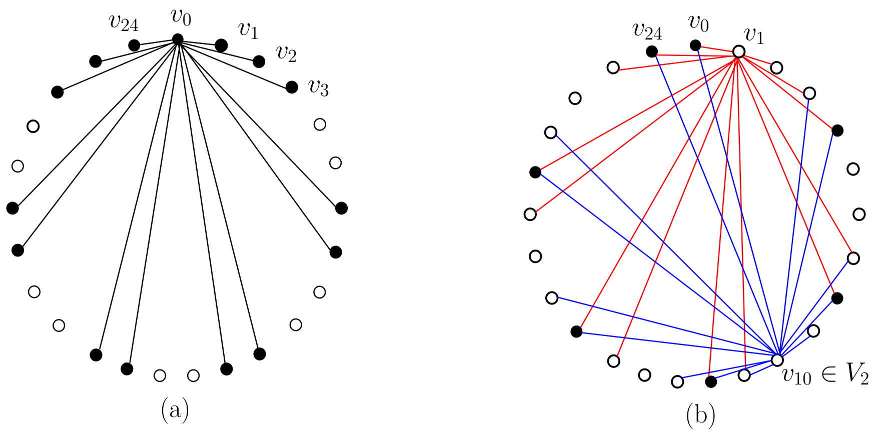

Figure 2: (a) Illustrating the regular construction of G_6. (b) Illustrating the neighbourhood of v_1 and some $v \in V_2$. Observe both neighbourhoods include V_0, the black vertices.

Regular construction: Let G_p denote the graph with $4p+1$ vertices $\{v_0, v_1, \dots, v_{4p}\}$ where the neighbourhood of each vertex is given by:

$$N(v_i) = \{v_{i-1}, v_{i+1}\} \cup \{v_{i+4j+2},\ v_{i+4j+3} \mid 0 \le j \le p{-}1\}$$

with indices taken modulo $4p+1$.

Figure 2(a) illustrates the regular neighbourhood structure for G_6. The vertices for a given G_p can be partitioned into 4 sets V_0, V_1, V_2, V_3 where each $V_i = \{v_t \mid t \equiv i \bmod 4\}$. Observe that v_s is not adjacent to v_t for any $v_s, v_t \in V_i$ except for the case when $\{v_s, v_t\} = \{v_0, v_{4p}\}$. This implies the following remark.

Remark 2.1 *Given G_p for $p \ge 2$, V_1, V_2, and V_3 are stable sets and the only edge in V_0 is between v_0 and v_{4p}.*

Lemma 2.2 *G_p is $2K_2$-free, for $p \ge 2$.*

Proof. Consider the vertices $\{v_0, v_1, \dots, v_{4k}\}$ for a given G_k, where $k \ge 2$, and recall its partition into V_0, V_1, V_2, V_3. Since G_k is regular, WLOG consider an edge (v_0, v_j) and suppose that it belongs to a $2K_2$ with edge (x, y). Observe that $N(v_0) = \{v_1, v_{4k}\} \cup V_2 \cup V_3$. By symmetry, we need only consider two cases for v_j: either $v_j = v_1$ or $v_j \in V_2$ (V_3 becomes V_2 in the reflection centered at v_0). In both cases $V_0 \subseteq N(v_j)$, as illustrated in Figure 2(b) for G_6. Thus, since x and y are both not adjacent to either v_0 or v_j, they must belong to V_1. From Remark 2.1, V_1 is a stable set, which contradicts the edge (x, y). Thus, v_0 does not belong to a $2K_2$. □

Since every P_5 contains a $2K_2$, we obtain the following corollary.

Corollary 2.3 *G_p is P_5-free, for $p \ge 2$.*

The following theorem proves that there an infinite number of 5-vertex-critical P_5-free graphs.

Theorem 2.4 *G_p is 5-vertex-critical, for all $p \ge 2$.*

Proof. Suppose G_p is 4-colorable. Observe that the vertices $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$ form a K_4 for any i (modulo $4p+1$). WLOG, assign colors 0, 1, 2, 3 to v_0, v_1, v_2, v_3 respectively. It is easy to see that each vertex, v_i for $i \in \{4, 5, \ldots, 4p-1\}$, must have color i taken modulo 4. However, the vertex v_{4p} is adjacent to a vertex of each of the four colors, a contradiction. Thus G_p is not 4-colorable.

From Remark 2.1, the vertices of G_p can be partitioned into 3 stable sets V_1, V_2, V_3 along with V_0 which consists of a single edge (v_0, v_{4p}). Thus, G_p can be 5-colored by assigning colors 1, 2, 3 to the stable sets V_1, V_2, V_3 respectively, and then coloring $V_0 - \{v_{4p}\}$ with color 0 and coloring v_{4p} with 5. Clearly, by removing v_{4p} from G_p, the resulting graph is 4-colorable. Thus G_p is 5-vertex-critical. □

Given a graph G, let $G \vee u$ denote the graph obtained from G by adding a new vertex u and adding all edges between u and the vertices of G. We say that $G \vee u$ is obtained from G by adding a universal vertex. Let $\mathsf{H}_{p,k}$ be defined recursively as follows for $p \geq 2$ and $k \geq 5$:

$$\mathsf{H}_{p,k} = \begin{cases} \mathsf{G}_p & \text{if } k = 5 \\ \mathsf{H}_{p,k-1} \vee u & \text{if } k > 5. \end{cases}$$

It is easy to verify that if G is k-vertex-critical, then $G \vee u$ is $(k+1)$-vertex-critical. Thus the following corollary follows from the previous Theorem.

Corollary 2.5 $\mathsf{H}_{p,k}$ *is k-vertex-critical, for all $p \geq 2$ and $k \geq 5$.*

Since G_p is $2K_2$-free, observe that each graph $\mathsf{H}_{p,k}$ is also $2K_2$-free (and hence P_5-free) because adding a universal vertex will never introduce a new $2K_2$. Thus, for a fixed $k \geq 5$, the set of all $\mathsf{H}_{p,k}$, where $p \geq 2$, is an infinite set of k-vertex-critical P_5-free graphs. Recall, however, our original question was to determine whether or not there was a finite number of k-critical P_5-graphs, for fixed $k \geq 5$. We introduce one more lemma before resolving this question.

Lemma 2.6 *Let $\mathcal{C}$ be a class of graphs. If $G \in \mathcal{C}$ is k-vertex-critical, then there exists a subgraph of G on the same set of vertices that is k-critical (w.r.t. $\mathcal{C}$).*

Proof. Suppose the Lemma is false. Choose a graph G with the fewest number of edges that is a counter example. Since G is not k-critical, by definition there is a non-empty subset of edges E such that $G - E$ is not $(k-1)$-colorable and belongs to $\mathcal{C}$. But clearly $G - E$ is also k-vertex-critical, and thus also a counter example to the Lemma. But this contradicts our original choice of G. □

Together, Lemma 2.6 and Corollary 2.5 establish the existence of an infinite set of k-critical P_5-free graphs for fixed $k \geq 5$. However, we do not have a precise construction of such a set. Through an exhaustive computer search, focusing on $k = 5$, we found:

- ▷ 5 unique 5-critical P_5-free graphs that are proper subgraphs of G_3,
- ▷ 3 unique 5-critical P_5-free graphs that are proper subgraphs of G_4,
- ▷ 1 unique 5-critical P_5-free graphs that is a proper subgraph of G_5, and
- ▷ for $6 \leq p \leq 25$, G_p is a 5-critical P_5-free graph.

Figure 3 illustrates three 5-critical P_5-free graphs that are subgraphs of G_3, G_4, and G_5 respectively. A formal description of these nine subgraphs are given in the Appendix.

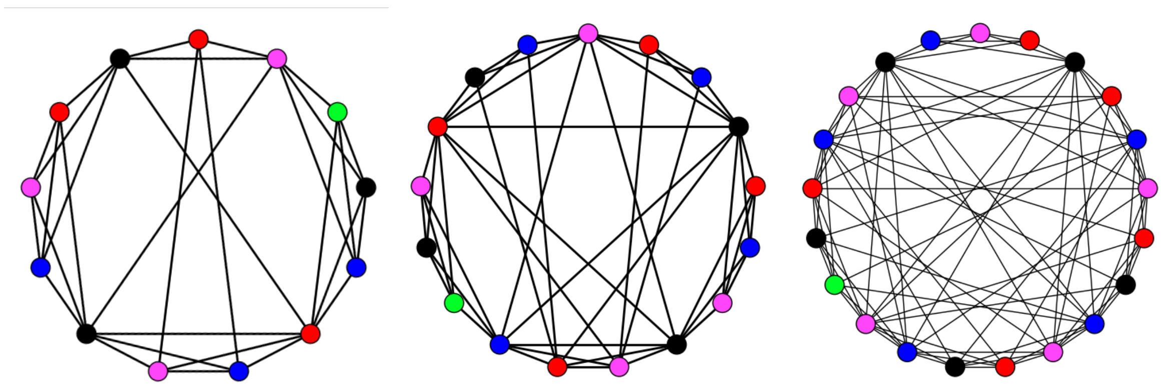

Figure 3: Three 5-critical P_5-free graphs on 13, 17, and 21 vertices. They are subgraphs of G_3, G_4, and G_5 respectively.

We now formally prove that G_p is a 5-critical P_5-free graph for all $p \geq 6$. Part of the proof relies on a computer aided test on two graphs with 23 vertices. In particular, given a set of forced edges E_F and one of our graphs G, we need to verify that G does not have a P_5-free subgraph that contains E_F. The function CHECK-P5 given in Algorithm 1 can be used for this verification. It is a simple recursive approach that considers whether or not a given edge e could belong to a possible P_5-free subgraph. If it is part of the subgraph, then it is added to the list of forced edges E_F. Otherwise it is removed from the graph. If the input graph to the function is P_5-free, it returns True. If there is a P_5 that consists of only forced edges in E_F, the function returns False. Observe that if neither of these conditions hold, then there exists an edge in the input graph, $E(G)$, that is not in E_F.

Algorithm 1 Test if G contains a P_5-free subgraph containing E_F

function CHECK-P5(G, E_F) **returns** Boolean
  if G is P_5-free **then return** True
  if G has an induced P_5 with all 4 edges in E_F **then return** False
  Choose $e \in E(G) \setminus E_F$
  return CHECK-P5($G, E_F \cup e$) **or** CHECK-P5($G \setminus e, E_F$)

Theorem 2.7 *G_p is a 5-critical P_5-free graph, for all $p \geq 6$.*

Proof. Suppose G_p is not a 5-critical P_5-free graph. Then there exists a non-empty subset of edges E' such that $\mathsf{G}_p \setminus E'$ is P_5-free and not 4-colorable. Recall that G_p has $n = 4p+1$ vertices. We refer to an edge (v_i, v_{i+d}) as a distance d edge, where indices are considered modulo n . Consider 4 cases for E' based on various distances:

Case 1: Suppose E' contains an edge of distance 1. WLOG let (v_{4p}, v_0) be such an edge. This implies that the vertices of $\mathsf{G}_p \setminus E'$ can be partitioned into 4 stable sets from Remark 2.1. Such a graph is 4-colorable, which is a contradiction. Thus, E' cannot contain an edge of distance 1.

Case 2: Suppose E' contains an edge of distance 2. WLOG let (v_1, v_3) be such an edge. Consider the subgraph H of G_p induced by $v_0, v_1, v_2, \ldots, v_{22}$ with this edge removed. Observe that

H is the same for each G_p where $p \geq 6$. Setting E_F to be all the edges of distance 1 in H, we run the algorithm CHECK-P5(H, E_F) given in Algorithm 1. It returns False, which proves that every subgraph of H including E_F contains a P_5. Thus, for any set of edges E' that includes (v_1, v_3) and no distance 1 edges, $\mathsf{G}_p \setminus E'$ contains a P_5, a contradiction. Thus, E' cannot contain an edge of distance 2.

Case 3: We apply the same steps as Case 2 replacing the edge (v_1, v_3) with (v_1, v_4) to prove that E' cannot contain an edge of distance 3.

Case 4: Suppose E' contains no edge of distance ≤ 3. Since E' is non-empty WLOG let (v_0, v_t) be an edge of *minimal distance* in E'. Since $t > 3$, it must be that $t \bmod 4$ is either 2 or 3. If $t \bmod 4 = 2$ then the path $v_0, v_1, v_4, v_t, v_{t+3}$ is an induced P_5 consisting of edges of distance less than t. If $t \bmod 4 = 3$ then the path $v_0, v_3, v_4, v_t, v_{t+1}$ is an induced P_5 consisting of edges of distance less than t. In both cases $\mathsf{G}_p \setminus E'$ contains a P_5, a contradiction.

Since the above four cases cover all eventualities, the Theorem is proved. □

It is interesting to note that for $k \geq 6$, G_k contains a C_5. In particular, one such induced cycle is given by the sequence $v_0, v_{10}, v_{4k-4}, v_5, v_{4k-9}, v_0$. In the next section we show that there are a finite number of 5-critical $\{P_5, C_5\}$-free graphs.

3 $\{P_5, C_5\}$-free graphs

In this section we describe an algorithm that verifies there are exactly eight 5-critical $\{P_5, C_5\}$-free graphs. We opt for a programmatic approach since a case-based proof similar to that given in [2] is far too tedious for these more complex graphs.

To begin, we consider a generic algorithm to exhaustively generate all k-critical graphs with respect to a class $\mathcal{C}$ that can be described by some forbidden subgraph characterization. Such an algorithm, EXTEND_ALL, is outlined in Algorithm 2. The algorithm takes as input a set of graphs $X_n \in \mathcal{C}$ on n vertices, that are $(k-1)$-colorable. For each graph G in X_n an isolated vertex is added and every edge combination involving the new vertex is considered. For each new graph in $\mathcal{C}$: if it is $(k-1)$-colorable, it is added to the set X_{n+1}; otherwise, if adding the new vertex increases the chromatic number to k while also being k-critical, then it is added to Y_{n+1}. The $(k-1)$-colorable graphs become the input for the next run of the algorithm. Before making the recursive call, isomorphisms are removed from X_{n+1} and Y_{n+1} using 'nauty'[1]. The initial call is EXTEND_ALL(X_1) where X_1 contains a single graph with one vertex, and Y_1 is initialized appropriately.

The algorithm terminates only if an input X_n is empty. In this case, the algorithm proves that $Y_1 \cup Y_2 \cup \cdots \cup Y_n$ are precisely the k-critical graphs with respect to the class $\mathcal{C}$.

For our purposes, we want to use the algorithm for $k = 5$ where $\mathcal{C}$ is the set of $\{P_5, C_5\}$-free graphs. The check if $G \in \mathcal{C}$ can be done simply by testing if any set of 5 vertices is a P_5 or C_5. The chromatic number of G can be determined by a simple recursive backtracking approach. As an optimization, if the chromatic number of G is k, any subsequent edge set (considered on line 6) that contains E' can be skipped since the resulting graph will not be k-critical (w.r.t. $\mathcal{C}$). To test if a graph G is k-critical, we first test if it is k-vertex-critical by considering the chromatic number

[1] Version 2.4 (r2) from `http://cs.anu.edu.au/~bdm/nauty/`

Algorithm 2 Extend all graphs in $X_n \in \mathcal{C}$

procedure EXTEND_ALL(X_n)
  $X_{n+1} \leftarrow \emptyset$
  $Y_{n+1} \leftarrow \emptyset$
  for each $G \in X_n$ **do**
    $V(G) \leftarrow V(G) \cup \{n+1\}$
    for each non-empty $E' \subseteq \{(v_0, v_{n+1}), (v_1, v_{n+1}), \ldots, (v_n, v_{n+1})\}$ **do**
      $E(G) \leftarrow E(G) \cup E'$
      if $G \in \mathcal{C}$ **then**
        if $\chi(G) = k$ **then**
          if G is k-critical **then** $Y_{n+1} \leftarrow Y_{n+1} \cup G$
        else $X_{n+1} \leftarrow X_{n+1} \cup G$
      $E(G) \leftarrow E(G) \backslash E'$
  REMOVE_ISOMORPHISMS(X_{n+1})
  REMOVE_ISOMORPHISMS(Y_{n+1})
  EXTEND_ALL(X_{n+1})

of $G - v$ for each vertex $v \in G$. If it is, then we consider all subsets of edges E'' such that $G - E''$ is $\{P_5, C_5\}$-free. If all such $G - E''$ are also $(k-1)$-colorable, then G is k-critical.

To optimize our search we apply the Strong Perfect Graph Theorem [3]. Since K_5 is a 5-critical $\{P_5, C_5\}$-free graph, this theorem implies that any other such graph must contain a C_k or its complement $\overline{C_k}$ as an induced subgraph for some odd $k \geq 5$. However:

- $\overline{C_5} = C_5$ is forbidden,
- each C_k contains a forbidden P_5, for odd $k \geq 5$,
- each $\overline{C_k}$ contains a K_5, for odd $k \geq 11$,
- $\overline{C_9}$ contains a proper subgraph that is 5-vertex-critical (the graph on nine vertices in Figure 4).

Thus, every other 5-critical $\{P_5, C_5\}$-free graph must contain a $\overline{C_7}$. Hence, as a starting point for our search, we set X_7 to contain the single graph $\overline{C_7}$ as the first input of the program.

Unfortunately, this algorithm will never terminate. For example, let the vertex set of $\overline{C_7}$ be $\{c_1, c_2, \ldots, c_7\}$. The program can extend the graph by adding a vertex, u_1, with $N(u_1) = N(c_1)$. This new graph, G', is $\{P_5, C_5\}$-free and 4-colorable. The graph G' can be extended further by adding a vertex, u_2, with $N(u_2) = N(c_1)$. Again, this new graph, G'', is $\{P_5, C_5\}$-free and 4-colorable. Adding such vertices can continue forever, and so X_n will never be empty. Thus, we consider some additional properties of k-vertex-critical graphs.

Lemma 3.1 *Let G be a graph with chromatic number k. If G contains two disjoint m-cliques $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ such that $N(a_i) \subseteq N(b_i)$ for all $1 \leq i \leq m$, then G is not k-vertex-critical.*

Proof. Suppose G is k-vertex-critical. Then, $G-A$ must be $(k-1)$-colorable. Apply such a $(k-1)$-coloring to the corresponding vertices of G and assign a_i the color of b_i. The result is a valid $(k-1)$-coloring, a contradiction. □

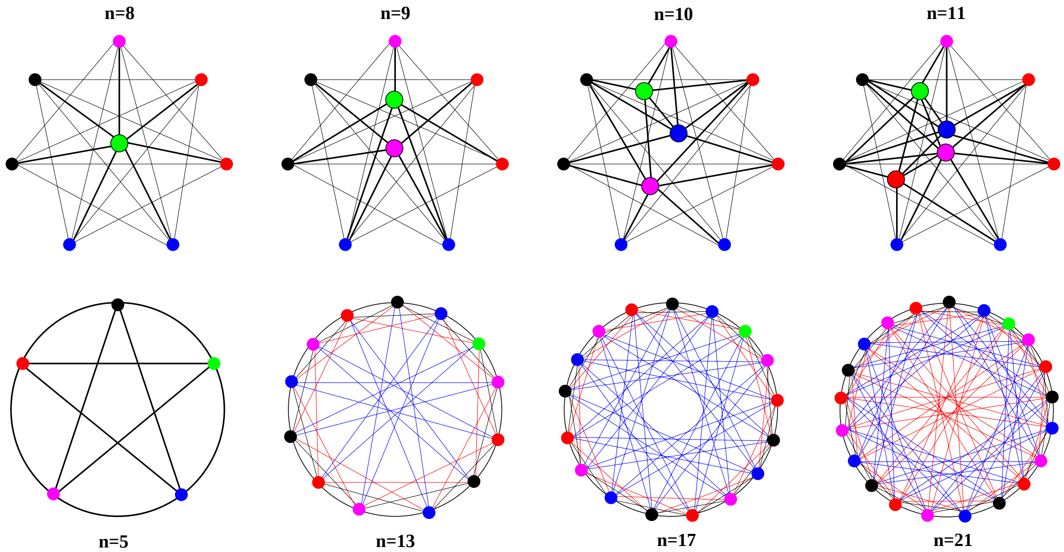

Figure 4: The eight 5-critical $\{P_5, C_5\}$ -free graphs.

We note that the case $m = 1$ is a well known folklore in graph coloring. We apply Lemma 3.1 to the exhaustive search as follows. First, we consider an application of $m = 1$: if there exists two vertices u and v in the current graph such that $N(u) \subseteq N(v)$, then in order for any extension of the graph to ever be k-vertex-critical, there must be some vertex w added that is adjacent to u but not v. WLOG, we can make this the new vertex added to the graph. If no such vertices exist, then we consider an application of the lemma when $m = 2$: if there exists two disjoint edges (u, u') and (v, v') in the current graph such that $N(u) \subseteq N(v)$ and $N(u') \subseteq N(v')$, then in order for any extension of the graph to ever be k-vertex-critical, there must be some vertex w added that is either adjacent to u but not v or adjacent to u' but not v'. WLOG, we can make this the new vertex added to the graph.

By applying these applications of Lemma 3.1, it is sufficient to force the exhaustive search to terminate (in about 2 minutes) when $X_{21} = \emptyset$ giving us the following result:

Theorem 3.2 *There are eight 5-critical $\{P_5, C_5\}$-free graphs.*

The eight $\{P_5, C_5\}$-free graphs are listed in Figure 4. Their formal descriptions are given in the Appendix. The three largest graphs found with $n = 13, 17, 21$ are isomorphic to G_3, G_4, and G_5 respectively.

This theorem implies that one can answer whether or not a $\{P_5, C_5\}$-free graph G is 4-colorable by testing whether or not G contains one of the 8 5-critical graphs as a subgraph. If the graph is not 4-colorable, this approach yields a polynomial time algorithm for providing a *no-certificate* of a minimal obstruction causing the graph to be non 4-colorable.

4 Open Problems

There are a number of interesting open problems related to this work:

1. Is there a generic infinite construction of k-critical P_5-free graphs when $k \geq 5$?
2. Other than the infinite set of 5-critical P_5-free graphs described in this paper, are there a finite number of other graphs that are 5-critical?
3. Is the 3-colorability question polynomial time solvable for P_t-free graphs, for any fixed t?
4. Is the 4-colorability question polynomial time solvable for P_6-free graphs?
5. Is the STABLE SET problem for $\{P_5, C_5\}$-free graphs NP-hard? An overview of this problem for $\{P_5, X\}$-free graphs for a variety of small graphs X is given in [11].

A Appendix

Edge listings for the 9 proper subgraphs of G_3, G_4 and G_5 that are 5-critical P_5-free graphs.

$n = 13$: (1,2),(1,4),(1,7),(1,12),(1,13),(2,3),(2,4),(2,5),(2,8),(2,9),(2,12),(2,13),(3,4),(3,6),(3,13),(4,5),(4,6),(4,7),(4,10),(4,11),
(5,6),(5,7),(5,12),(6,7),(6,8),(6,9),(6,12),(6,13),(7,8),(7,9),(7,13),(8,9),(8,10),(8,11),(9,10),(9,11),(10,11),(10,12),
(10,13),(11,12),(11,13),(12,13)

$n = 13$: (1,2),(1,4),(1,7),(1,8),(1,12),(1,13),(2,3),(2,4),(2,8),(2,9),(2,12),(3,4),(3,5),(3,6),(3,9),(3,13),(4,5),(4,6),(4,7),(4,10),
(4,11),(5,6),(5,7),(5,8),(5,12),(6,7),(6,8),(6,12),(7,8),(7,9),(7,13),(8,9),(8,10),(8,11),(9,10),(9,11),(9,12),(10,11),
(10,12),(10,13),(11,12),(11,13),(12,13)

$n = 13$: (1,2),(1,7),(1,12),(1,13),(2,3),(2,4),(2,5),(2,8),(2,12),(2,13),(3,4),(3,5),(3,6),(4,5),(4,6),(5,6),(6,7),(6,8),(6,12),
(7,8),(7,13),(8,9),(8,10),(8,11),(9,10),(9,11),(9,12),(10,11),(10,12),(11,12),(12,13)

$n = 13$: (1,2),(1,7),(1,12),(1,13),(2,3),(2,4),(2,8),(2,12),(2,13),(3,4),(3,5),(3,6),(4,5),(4,6),(5,6),(5,7),(5,8),(5,12),(6,7),
(6,8),(6,12),(7,8),(7,13),(8,9),(8,10),(8,11),(9,10),(9,11),(9,12),(10,11),(10,12),(11,12),(12,13)

$n = 13$: (1,2),(1,7),(1,12),(1,13),(2,3),(2,4),(2,5),(2,8),(2,9),(2,12),(2,13),(3,4),(3,5),(3,6),(4,5),(4,6),(5,6),(6,7),(6,8),(6,9),
(6,12),(6,13),(7,8),(7,9),(8,9),(8,10),(8,11),(9,10),(9,11),(10,11),(10,12),(10,13),(11,12),(11,13),(12,13)

$n = 17$: (1,2),(1,3),(1,7),(1,11),(1,15),(1,16),(1,17),(2,3),(2,4),(2,5),(2,8),(2,9),(2,12),(2,16),(2,17),(3,4),(3,5),(3,6),(3,10),
(3,13),(3,14),(3,17),(4,5),(4,6),(4,7),(4,11),(4,15),(5,6),(5,7),(5,11),(5,15),(6,7),(6,8),(6,9),(6,12),(6,16),(7,8),(7,9),
(7,10),(7,13),(7,14),(7,17),(8,9),(8,10),(8,11),(8,15),(9,10),(9,11),(9,15),(10,11),(10,12),(10,16),(11,12),(11,13),
(11,14),(11,17),(12,13),(12,14),(12,15),(13,14),(13,15),(13,16),(14,15),(14,16),(15,16),(15,17),(16,17)

$n = 17$: (1,2),(1,15),(1,16),(1,17),(2,3),(2,4),(2,5),(2,8),(2,12),(2,13),(2,16),(2,17),(3,4),(3,5),(3,14),(4,5),(4,14),(5,6),(5,7),
(5,8),(5,12),(5,15),(6,7),(6,8),(6,13),(7,8),(7,13),(8,9),(8,10),(8,11),(8,14),(8,15),(9,10),(9,11),(9,12),(10,11),(10,12),
(11,12),(12,13),(12,14),(12,15),(13,14),(13,15),(14,15),(15,16),(15,17),(16,17)

$n = 17$: (1,2),(1,15),(1,16),(1,17),(2,3),(2,4),(2,8),(2,9),(2,12),(2,13),(2,16),(2,17),(3,4),(3,9),(3,14),(4,5),(4,6),(4,7),(4,10),
(4,11),(4,14),(4,15),(5,6),(5,7),(5,8),(6,7),(6,8),(7,8),(8,9),(8,10),(8,11),(8,14),(8,15),(9,10),(9,11),(9,15),(10,11),
(10,12),(10,13),(11,12),(11,13),(12,13),(12,14),(12,15),(13,14),(13,15),(14,15),(15,16),(15,17),(16,17)

$n = 21$: (1,2),(1,19),(1,20),(1,21),(2,3),(2,4),(2,5),(2,8),(2,9),(2,12),(2,13),(2,16),(2,17),(2,20),(2,21),(3,4),(3,5),(3,9),(3,13),
(3,17),(3,18),(4,5),(4,6),(4,7),(4,10),(4,11),(4,14),(4,15),(4,18),(4,19),(5,6),(5,7),(5,8),(5,12),(5,16),(5,19),(6,7),
(6,8),(6,9),(6,13),(6,17),(7,8),(7,9),(7,13),(7,17),(8,9),(8,10),(8,11),(8,14),(8,15),(8,18),(8,19),(9,10),(9,11),(9,12),
(9,16),(9,19),(10,11),(10,12),(10,13),(10,17),(11,12),(11,13),(11,17),(12,13),(12,14),(12,15),(12,18),(12,19),
(13,14),(13,15),(13,16),(13,19),(14,15),(14,16),(14,17),(15,16),(15,17),(16,17),(16,18),(16,19),(17,18),(17,19),
(18,19),(19,20),(19,21),(20,21)

Edge listings for the eight 5-critical $\{P_5, C_5\}$-free graphs.

$n = 5$: (1,2),(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,4),(3,5),(4,5)

$n = 8$: (1,3),(1,4),(1,5),(1,6),(1,8),(2,4),(2,5),(2,6),(2,7),(2,8),(3,5),(3,6),(3,7),(3,8),(4,6),(4,7),(4,8),(5,7),(5,8),(6,8),(7,8)

$n = 9$: (1,3),(1,4),(1,5),(1,6),(1,8),(1,9),(2,4),(2,5),(2,6),(2,7),(2,8),(2,9),(3,5),(3,6),(3,7),(3,8),(3,9),(4,6),(4,7),(4,8),(5,7),(5,9),(6,8),(7,9)

$n = 10$: (1,3),(1,4),(1,5),(1,6),(1,8),(1,9),(1,10),(2,4),(2,5),(2,6),(2,7),(2,8),(2,9),(3,5),(3,6),(3,7),(3,8),(3,9),(3,10),(4,6),(4,7),(4,9),(4,10),(5,7),(5,10),(6,10),(7,9),(7,10),(8,9),(8,10)

$n = 11$: (1,3),(1,4),(1,5),(1,6),(1,8),(1,9),(1,10),(1,11),(2,4),(2,5),(2,6),(2,7),(2,8),(2,9),(2,11),(3,5),(3,6),(3,7),(3,8),(3,11),(4,6),(4,7),(4,8),(4,9),(5,7),(5,8),(5,9),(6,9),(6,10),(7,9),(7,10),(8,10),(8,11),(9,10),(9,11),(10,11)

$n = 13$: (1,3),(1,4),(1,5),(1,6),(1,8),(1,10),(1,11),(1,12),(2,4),(2,5),(2,6),(2,7),(2,8),(2,9),(2,10),(2,11),(3,5),(3,6),(3,7),(3,8),(3,9),(3,10),(3,11),(4,6),(4,7),(4,8),(4,9),(4,10),(4,13),(5,7),(5,8),(5,9),(5,12),(5,13),(6,9),(6,11),(6,12),(6,13),(7,10),(7,11),(7,12),(7,13),(8,11),(8,12),(8,13),(9,10),(9,11),(9,12),(10,12),(10,13),(11,13),(12,13)

$n = 17$: (1,3),(1,4),(1,5),(1,6),(1,8),(1,10),(1,11),(1,13),(1,14),(1,15),(2,4),(2,5),(2,6),(2,7),(2,8),(2,10),(2,11),(2,13),(2,14),(2,15),(3,5),(3,6),(3,7),(3,8),(3,9),(3,10),(3,13),(3,14),(3,17),(4,6),(4,7),(4,8),(4,9),(4,10),(4,13),(4,14),(4,17),(5,7),(5,8),(5,9),(5,12),(5,13),(5,16),(5,17),(6,9),(6,11),(6,12),(6,15),(6,16),(6,17),(7,10),(7,11),(7,12),(7,14),(7,15),(7,16),(8,11),(8,12),(8,14),(8,15),(8,16),(9,10),(9,11),(9,12),(9,14),(9,15),(9,16),(10,12),(10,15),(10,16),(10,17),(11,12),(11,13),(11,16),(11,17),(12,13),(12,14),(12,15),(13,15),(13,16),(13,17),(14,16),(14,17),(15,17),(16,17)

$n = 21$: (1,3),(1,4),(1,5),(1,6),(1,8),(1,10),(1,11),(1,14),(1,15),(1,19),(1,20),(1,21),(2,4),(2,5),(2,6),(2,7),(2,8),(2,10),(2,11),(2,14),(2,15),(2,16),(2,20),(2,21),(3,5),(3,6),(3,7),(3,8),(3,9),(3,10),(3,15),(3,16),(3,17),(3,18),(3,21),(4,6),(4,7),(4,8),(4,9),(4,10),(4,15),(4,16),(4,17),(4,18),(4,21),(5,7),(5,8),(5,9),(5,12),(5,13),(5,16),(5,17),(5,18),(5,19),(6,9),(6,11),(6,12),(6,13),(6,14),(6,18),(6,19),(6,20),(7,10),(7,11),(7,12),(7,13),(7,14),(7,15),(7,19),(7,20),(8,11),(8,12),(8,13),(8,14),(8,18),(8,19),(8,20),(9,10),(9,11),(9,12),(9,14),(9,15),(9,19),(9,20),(9,21),(10,12),(10,13),(10,16),(10,17),(10,18),(10,19),(11,13),(11,15),(11,16),(11,17),(11,18),(11,21),(12,14),(12,15),(12,16),(12,17),(12,20),(12,21),(13,14),(13,15),(13,16),(13,17),(13,20),(13,21),(14,16),(14,17),(14,18),(14,19),(15,17),(15,18),(15,19),(16,18),(16,19),(16,20),(17,19),(17,20),(17,21),(18,20),(18,21),(19,21),(20,21)

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