A Hamilton Path for the Sigma-Tau Problem

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Abstract

Nijenhuis and Wilf asked the following question in their *Combinatorial Algorithms* textbook from 1975: Can the permutations of $\{1, 2, \ldots, n\}$ be ordered so that each permutation is transformed into the next by applying either the operation σ, a rotation to the left, or τ, a transposition of the first two symbols? Knuth rated the challenge of finding a cyclic solution for odd n (cycles do not exist for even $n > 2$) at 48/50 in *The Art of Computer Programming*, which makes it Volume 4's hardest open problem since the 'middle levels' problem was sovled by Mütze. In this paper we solve the 40 year-old question by Nijenhuis and Wilf, by providing a simple successor rule to generate each successive permutation. We also present insights into how our solution can be modified to find a Hamilton cycle for odd n.

1 Introduction

The efficient ordering and generation of permutations has a long and interesting history with surveys by Sedgewick in the 1970s [8], Savage in the 1990s [7], and more recently by Knuth [1]. However there has remained one long-standing open problem in the area. This problem was first articulated in 1975 in first edition of the *Combinatorial Algorithms* textbook by Nijenhuis and Wilf [3].

> **6.** The symmetric group S_n is generated by just *two* elements
>
> $$t : 1 \to 2;\ 2 \to 1;\ 3 \to 3;\ \ldots\ ;\ n \to n$$
>
> $$u : 1 \to 2 \to 3 \to \cdots \to n \to 1$$
>
> Sequence the 24 elements of S_4 so that each is obtained from its predecessor by either t or u. Can you do S_5? [7, 23]

To formally state the general version of this problem, consider the following two operations on a permutation $\pi = p_1 p_2 \cdots p_n$:

$$\sigma(\pi) = p_2 p_3 \cdots p_n p_1 \quad \text{and} \quad \tau(\pi) = p_2 p_1 p_3 p_4 \cdots p_n.$$

The operation σ (u in the original problem) rotates a permutation one position to the left and τ (t in the original problem) transposes the first two elements. The *Sigma-Tau* graph $\mathcal{G}_n$ is a directed graph

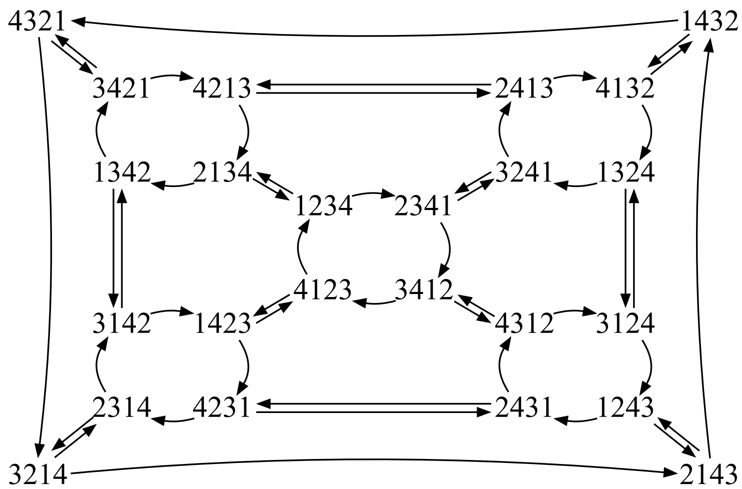

Figure 1: The Sigma-Tau graph $\mathcal{G}_4$ where the straight arcs are τ edges and the curved arcs are σ edges.

where the vertices are the permutations of $\{1, 2, \ldots, n\}$. There is a directed edge from π_1 to π_2 if and only if $\pi_2 = \sigma(\pi_1)$ or $\pi_2 = \tau(\pi_1)$. The Sigma-Tau graph $\mathcal{G}_4$ is illustrated in Figure 1.

The generalized statement of Nijenhuis and Wilf's original question is as follows.

Sigma-Tau Path Problem For $n > 1$, does there exist is a Hamilton path in $\mathcal{G}_n$?

In this paper we solve this problem, answering the question in the affirmative by providing an explicit Hamilton path construction.

The cycle version of this problem is known to have no solution for even $n > 2$. This follows from a more general Hamilton cycle condition by Rankin [5] (see Swan [9] for a simplified proof).[1] The following is a Hamilton cycle in $\mathcal{G}_3$:

$$231,\ 312,\ 132,\ 321,\ 213,\ 123.$$

It applies the operations $\sigma, \tau, \sigma, \sigma, \tau$ followed by σ to return to the first permutation. For odd $n \geq 3$, the problem of finding a Hamilton cycle was stated in Knuth's *The Art of Computer Programming*. It was ranked 48/50 making it the hardest open problem in the fascicle version of Volume 4 [1] since the middle levels problem which was rated 49/50 was recently solved by Mütze [2].

71. [*48*] Does the Cayley graph with generators $\sigma = (1\,2\,\ldots\,n)$ and $\tau = (1\,2)$ have a Hamiltonian cycle whenever $n \geq 3$ is odd?

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[1] An error in [4] stated that a Hamilton path cycle was possible for $n = 5$. Ruskey, Jiang, and Weston [6] corrected this error by finding all five non-isomorphic cyclic orders for $n = 5$.

In the following section, we present some necessary definitions and notation along with some preliminary results. In Section 3 we describe how $\mathcal{G}_n$ can be partitioned into 2 cycles, and then ultimately provide a construction for a Hamilton path in $\mathcal{G}_n$. In Section 4 we give insights into the construction of a Hamilton cycle in $\mathcal{G}_n$, for odd n. The Appendix contains a C implementation for the construction of a Hamilton path in $\mathcal{G}_n$. The Hamilton path and cycle rules discussed in this article also appear in an unpublished manuscript [10]. This article gives a more accessible proof for the path case.

2 Seeds and Flowers

In this section, we provide the necessary definitions and notation used to prove our main result. For the rest of this paper we assume that n is fixed to be greater than 3.

Let $\mathbf{P}$ denote the set of all permutations of $\{1, 2, \ldots, n\}$. By removing all τ edges in the Sigma-Tau graph $\mathcal{G}_n$, the permutations are partitioned into $(n-1)!$ cycles. The n permutations within each cycle form an equivalence class under rotation. Let the lexicographically largest permutation of each cycle be its representative, and we call such a permutation a *cyclic permutation*. Let $\mathbf{cycle}(\pi)$ denote the set of n permutations rotationally equivalent to π. When representing a permutation, the last symbol can be inferred from the first $n-1$ symbols. A *shorthand permutation* is a length $n-1$ prefix of some permutation. A *seed* is a shorthand permutation $\boldsymbol{s} = s_1 s_2 \cdots s_{n-1}$ where $s_1 = n$ and the missing symbol x is s_2+1, unless $s_2 = n-1$ in which case $x = 1$. There are $(n-1)(n-3)!$ seeds. Given a seed $\boldsymbol{s}$ with missing symbol x, the *flower of $\boldsymbol{s}$*, denoted $\mathbf{flower}(\boldsymbol{s})$, is the set of all $n-1$ cyclic permutations that can be obtained by inserting x after a symbol in $\boldsymbol{s}$. Given a seed $\boldsymbol{s}$, let $\mathbf{perms}(\boldsymbol{s})$ denote $\bigcup_{\pi \in \mathbf{flower}(\boldsymbol{s})} \mathbf{cycle}(\pi)$. If $\mathbf{S}$ is a set of seeds, let $\mathbf{perms}(\mathbf{S}) = \bigcup_{\boldsymbol{s} \in \mathbf{S}} \mathbf{perms}(\boldsymbol{s})$.

Example 1 When $n = 5$ the $4 \cdot 2! = 8$ seeds are:

$$5432,\ 5423,\ 5321,\ 5312,\ 5241,\ 5214,\ 5143,\ 5134.$$

The flower of seed 5321 is $\mathbf{flower}(5321) = \{5\mathbf{4}321,\ 53\mathbf{4}21,\ 532\mathbf{4}1,\ 5321\mathbf{4}\}$.

$$\begin{aligned} \mathbf{perms}(5321) \;=\; & 54321, 43215, 32154, 21543, 15432, \\ & 53421, 34215, 42153, 21534, 15342, \\ & 53241, 32415, 24153, 41532, 15324, \\ & 53214, 32145, 21453, 14532, 45321. \end{aligned}$$

For the remainder of this paper, arithmetic on the symbols is performed $\bmod\ n-1$, where $n \equiv 1$ and $0 \equiv n-1$. Observe that each cyclic permutation $\pi = p_1 p_2 \cdots p_n$ belongs to at least one flower. In particular the flower of the seed obtained by removing p_2+1 from π contains π. This leads to the following lemma.

Lemma 2.1 *Let* $\mathbf{S}$ *be the set of all seeds. Then* $\mathbf{perms}(\mathbf{S}) = \mathbf{P}$.

Let $G = (V, E)$ be a directed graph with vertex set V and edge set E. Given a subset V' of V, let $G[V']$ denote the subgraph of G induced by V'. A vertex v is a *neighbour* of a vertex u if

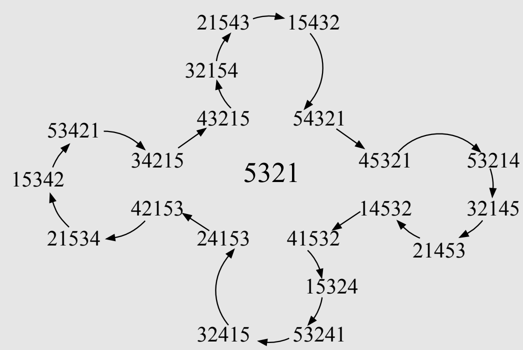
$(u, v) \in E$. A *successor rule* on a directed graph is a function that maps each vertex onto one of its neighbours. In the following, we define a successor rule that can be used to construct a Hamilton cycle in $\mathcal{G}_n[\mathbf{perms}(\boldsymbol{s})]$ for an arbitrary seed $\boldsymbol{s}$.

Consider a seed $\boldsymbol{s} = s_1 s_2 \cdots s_{n-1}$ with missing symbol x. For $1 \leq j < n$, consider the cyclic permutation obtained by inserting x after s_j. Let π_j denote the rotation of this permutation with x in the second position. Define a successor rule f for $\mathcal{G}_n[\mathbf{perms}(\boldsymbol{s})]$ as follows:

$$f(\pi) = \begin{cases} \tau(\pi) & \text{if } \pi = \pi_j \text{ for some } 1 \leq j < n; \\ \sigma(\pi) & \text{otherwise.} \end{cases}$$

Example 2 Consider seed $\boldsymbol{s} = 5321$ with missing symbol $x = 4$. Repeated application of the successor rule f constructs the following Hamilton cycle in $\mathcal{G}_5(\mathbf{perms}(5321))$:

45321, 53214, 32145, 21453, 1**4**532 $= \pi_4$,
41532, 15324, 53241, 32415, 2**4**153 $= \pi_3$,
42153, 21534, 15342, 53421, 3**4**215 $= \pi_2$,
43215, 32154, 21543, 15432, 5**4**321 $= \pi_1$.



The five permutations in each row are equivalent under rotation. A τ transition is applied to move between the equivalence classes when the second symbol is the missing symbol $x = 4$.

Note that $\tau(\pi_j) = \sigma(\pi_{j-1})$, where $\tau(\pi_1) = \sigma(\pi_{n-1})$. Thus, repeated application of f constructs a Hamilton cycle, denoted $Ham(\boldsymbol{s})$, in $\mathcal{G}_n[\mathbf{perms}(\boldsymbol{s})]$ as follows:

$$\begin{array}{rlllll} Ham(\boldsymbol{s}) = & \sigma(\pi_{n-1}), & \sigma^2(\pi_{n-1}), & \ldots, & \sigma^{n-1}(\pi_{n-1}), & \pi_{n-1}, \\ & \sigma(\pi_{n-2}), & \sigma^2(\pi_{n-2}), & \ldots, & \sigma^{n-1}(\pi_{n-2}), & \pi_{n-2}, \\ & \sigma(\pi_{n-3}), & \sigma^2(\pi_{n-3}), & \ldots, & \sigma^{n-1}(\pi_{n-3}), & \pi_{n-3}, \\ & \cdots & \cdots & \cdots & \cdots & \cdots \\ & \sigma(\pi_1), & \sigma^2(\pi_1), & \ldots, & \sigma^{n-1}(\pi_1), & \pi_1. \end{array}$$

Here σ^j denotes $\sigma^{j-1}(\sigma(j))$ for $j > 1$.

Lemma 2.2 *For any seed* $\boldsymbol{s}$*, repeated application of the successor rule* f *constructs a Hamilton cycle in* $\mathcal{G}_n[\mathbf{perms}(\boldsymbol{s})]$ *using exactly* $n - 1$ τ *edges.*

2.1 A Tree-like Structure of Seeds

The seeds of $\mathbf{P}$ can be ordered into an *almost* tree-like structure (a unicyclic graph). Consider a seed $\boldsymbol{s} = s_1 s_2 \cdots s_{n-1}$ with missing symbol x. Define the *parent* of $\boldsymbol{s}$, denoted $parent(\boldsymbol{s})$, to be the seed obtained by removing $x+1$ from $s_1 x s_2 \cdots s_{n-1}$, except when $x = n-1$ the symbol 1 is removed. The *decreasing subsequence* of a seed $\boldsymbol{s}$ is the longest prefix of $(s_2-1)(s_2-2) \cdots (s_2-n+3)$ that appears as a subsequence in $s_1, s_2, \ldots, s_{n-1}$. The *level* of a seed $\boldsymbol{s}$ is $(n-3)$ minus the length of its decreasing subsequence.

Example 3 Consider the seed $\boldsymbol{s} = 92518476$. Its decreasing subsequence is 1876 and thus is at level $(9-3)-4=2$. Also, $parent(\boldsymbol{s}) = 93251876$ has decreasing subsequence 21876 and it is at level 1.

The *hub*, denoted hub, is the set of $n-1$ seeds at level 0. An example of the hub and parent structure for $n=6$ is illustrated in Figure 2.

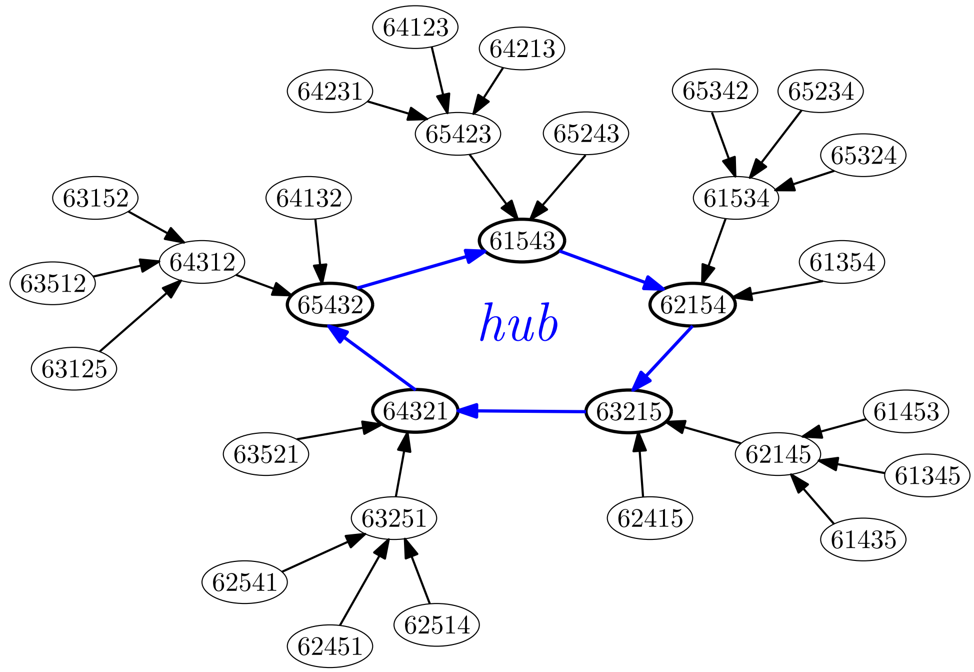


Figure 2: The seeds for $n=6$ illustrating the parent structure and hub.

Lemma 2.3 *If $\boldsymbol{s}$ is a seed at level $\ell > 0$, then $parent(s)$ is at level $\ell-1$.*

Proof. If $\boldsymbol{s} = s_1 s_2 \cdots s_{n-1}$ is a seed with missing symbol x at level $\ell > 0$, then $x = s_2+1$ and its decreasing subsequence has length $d < n-3$. Thus $x+1 = s_2-n-3$ is not part of its decreasing subsequence. The parent of s, which is the permutation $s_1 x s_2 \cdots s_{n-1}$ with $x+1$ removed, has decreasing subsequence of length $d+1$. Thus, $parent(s)$ is at level $\ell-1$. $\square$

Lemma 2.4 *Let $\boldsymbol{s}_1 = s_1 s_2 \cdots s_{n-1}$ be a seed with missing symbol x and let $\boldsymbol{s}_2$ be a seed not equal to $\boldsymbol{s}_1$. If $\boldsymbol{s}_2 = parent(\boldsymbol{s}_1)$ then $\mathbf{flower}(\boldsymbol{s}_1) \cap \mathbf{flower}(\boldsymbol{s}_2) = \{s_1 x s_2 \cdots s_{n-1}\}$. If $\boldsymbol{s}_2 \neq parent(\boldsymbol{s}_1)$ and $\boldsymbol{s}_1 \neq parent(\boldsymbol{s}_2)$ then $\mathbf{flower}(\boldsymbol{s}_1) \cap \mathbf{flower}(\boldsymbol{s}_2) = \emptyset$.*

Proof. Suppose $\boldsymbol{s}_2 = parent(\boldsymbol{s}_1)$. From the definition of parent, $s_1 x s_2 \cdots s_{n-1}$ is in $\mathbf{flower}(\boldsymbol{s}_1) \cap \mathbf{flower}(\boldsymbol{s}_2)$. Every other cyclic permutation in $\mathbf{flower}(\boldsymbol{s}_1)$ starts with $s_1 s_2$, where $s_2 = x-1$, and therefore clearly is not in $\mathbf{flower}(\boldsymbol{s}_2)$. Thus $\mathbf{flower}(\boldsymbol{s}_1) \cap \mathbf{flower}(\boldsymbol{s}_2) = \{s_1 x s_2 \cdots s_{n-1}\}$. Now suppose that $\boldsymbol{s}_2 \neq parent(\boldsymbol{s}_1)$ and $\boldsymbol{s}_1 \neq parent(\boldsymbol{s}_2)$ and that $\mathbf{flower}(\boldsymbol{s}_1) \cap \mathbf{flower}(\boldsymbol{s}_2)$ is not empty. Then $\mathbf{flower}(\boldsymbol{s}_1) \cap \mathbf{flower}(\boldsymbol{s}_2)$ must contain some cyclic permutation $\pi = s_1 s_2 \cdots s_j x s_{j+1} \cdots s_{n-1}$ where $2 < j \leq n$. Note that if $j = 1$ then $\boldsymbol{s}_2 = parent(\boldsymbol{s}_1)$. By removing any symbol from π except x or s_2, the resulting shorthand permutation is not seed, by its definition. However, if removing s_2 is a seed, then $\boldsymbol{s}_1 = parent(\boldsymbol{s}_2)$, a contradiction. Thus in this case $\mathbf{flower}(\boldsymbol{s}_1) \cap \mathbf{flower}(\boldsymbol{s}_2) = \emptyset$. ☐

3 Constructing a Hamilton Path in $\mathcal{G}_n$

In this section, we show that the following successor rule partitions $\mathcal{G}_n$ into two cycles. Then by making a small modification, we present a successor rule that constructs a Hamilton path in $\mathcal{G}_n$.

2-cycle successor rule.

Let $\mathbf{S}$ be a subset of permutations closed under σ. Let $\pi = p_1 p_2 \cdots p_n$ be a permutation and let r be the symbol to the right of n when π is considered cyclicly and skipping over p_2. Define the successor rule $next_{\mathbf{S}}$ on the induced subgraph $\mathcal{G}_n[\mathbf{S}]$ as follows:

$$next_{\mathbf{S}}(\pi) = \begin{cases} \tau(\pi) & \text{if } (r, p_2) \in \{(1,2), (2,3), \ldots, (n{-}2, n{-}1), (n{-}1, 1)\} \text{ and } \tau(\pi) \in \mathbf{S}; \\ \sigma(\pi) & \text{otherwise.} \end{cases}$$

Observe that $next_{\mathbf{S}}$ is equivalent to f when $\mathbf{S} = \mathbf{perms}(\boldsymbol{s})$ for some seed $\boldsymbol{s}$. This observation is important when considering the following results. To simplify our notation, we let $next$ denote $next_{\mathbf{S}}$ when $\mathbf{S} = \mathbf{P}$.

Lemma 3.1 *For $n > 3$, there are $(n{-}1)(n{-}1)(n{-}3)!$ permutations π in $\mathbf{P}$ such that $next(\pi) = \tau(\pi)$.*

Proof. The set of permutations $\pi = p_1 p_2 \cdots p_n$ such that $next(\pi) = \tau(\pi)$ can be partitioned by the position of n. By the conditions required for $\tau(\pi)$, $p_2 \neq n$. Thus, there are $(n{-}1)$ ways to place n. For each such placement, there are $(n{-}1)$ pairs of values possible for (r, p_2). This leaves $(n-3)!$ ways to place the remaining $n-3$ symbols. ☐

Lemma 3.2 *If $next(\pi) = \tau(\pi)$ for some permutation π, then there is a unique seed $\boldsymbol{s}$ such that both π and $\tau(\pi)$ are in $\mathbf{flower}(\boldsymbol{s})$.*

Proof. Given a seed $\boldsymbol{s}$, observe that each τ edge in $Ham(\boldsymbol{s})$ is between permutations π and $\tau(\pi)$ from two different cyclic permutations. Thus by Lemma 2.4, $\boldsymbol{s}$ is the unique seed such that both π and $\tau(\pi)$ are in $\mathbf{flower}(\boldsymbol{s})$. Summing over all seeds, this accounts for $(n-1)(n-1)(n-3)!$ τ edges in $\mathcal{G}_n$, which accounts for all τ edges from Lemma 3.1. ☐

The following outlines the two major steps required to prove that the successor rule $next$ partitions $\mathcal{G}_n$ into two cycles.

1. Show $next_{\mathbf{S}}$ partitions $\mathcal{G}_n[\mathbf{perms}(hub)]$ into two cycles C_1 and C_2.
2. Inductively grow C_1 by adding the permutations $\mathbf{perms}(\boldsymbol{s})$ one seed at a time.

For the first step, observe that the seeds in the hub can be ordered as follows:

$$\begin{array}{rcl} \boldsymbol{s}_1 & = & n(n-1)\cdots 2, \\ \boldsymbol{s}_2 & = & n1(n-1)(n-2)\cdots 3, \\ \boldsymbol{s}_3 & = & n21(n-1)(n-2)\cdots 4, \\ \cdots & \cdots & \cdots \\ \boldsymbol{s}_{n-1} & = & n(n-2)(n-3)\cdots 1. \end{array}$$

As an example, for $n = 5$,

$$\boldsymbol{s}_1 = 5432, \quad \boldsymbol{s}_2 = 5143, \quad \boldsymbol{s}_3 = 5214 \quad \boldsymbol{s}_4 = 5321.$$

For each seed $\boldsymbol{s}_j$, consider the last two permutations $\{q'_j, q_j\} = \{\sigma^{n-1}(\pi_1), \pi_1\}$ in $Ham(\boldsymbol{s}_j)$. Let $\mathbf{Q} = \bigcup_{1 \le j < n} \{q'_j, q_j\}$. When $n = 5$ we have

$$\mathbf{Q} = \{25143, 51432\} \cup \{35214, 52143\} \cup \{45321, 53214\} \cup \{15432, 54321\}.$$

Figure 3 illustrates how $next_{\mathbf{S}}$ partitions $\mathcal{G}_n[\mathbf{S}]$ into two cycles for $n = 5$, where $\mathbf{S} = \mathbf{perms}(hub)$.

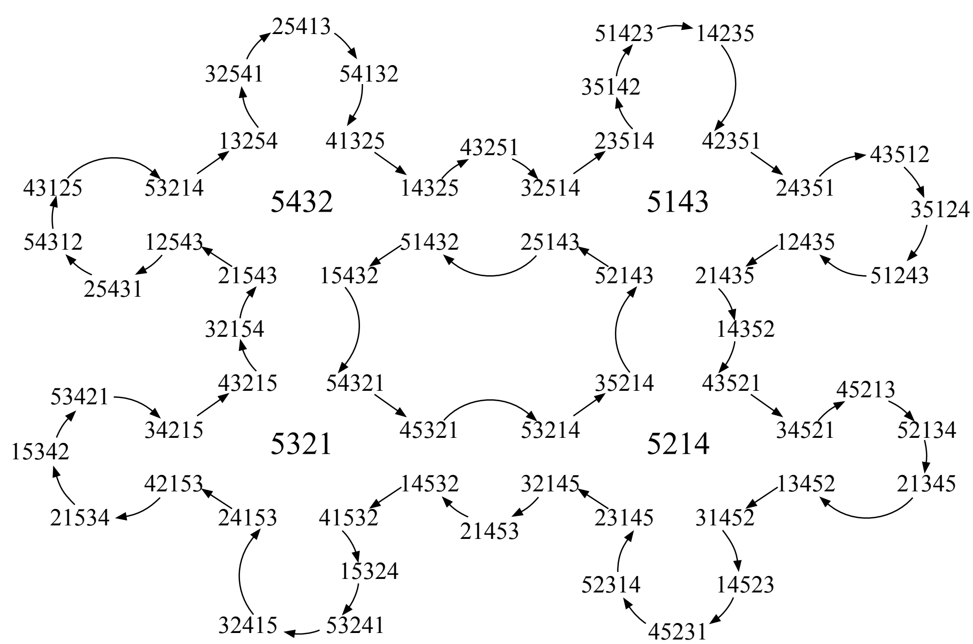


Figure 3: An illustration of how the successor rule $next_{\mathbf{S}}$ partitions $\mathcal{G}_5[\mathbf{S}]$ into two cycles, where $\mathbf{S} = \mathbf{perms}(hub)$. Note the inner cycle for $\mathbf{Q}$ given by 25143, 51432, 15432, 54321, 45321, 53214, 35214, 52143.

Lemma 3.3 *Let* $\mathbf{S} = \mathbf{perms}(hub)$*. Then* $next_{\mathbf{S}}$ *partitions* $\mathcal{G}_n[\mathbf{S}]$ *into two cycles, one for* $\mathbf{Q}$ *and one for* $\mathbf{S} - \mathbf{Q}$*.*

Proof. The following table illustrates that starting from permutation $n(n{-}1)\cdots 1$ and repeatedly applying $next_{\mathbf{S}}$ we obtain a Hamilton cycle in $\mathcal{G}_n[\mathbf{Q}]$. The cycle of permutations corresponds to the first column.

$\boldsymbol{\pi}$	$(\boldsymbol{r}, \boldsymbol{p_2})$	$\boldsymbol{next_{\mathbf{S}}(\pi)}$
$\boldsymbol{n}(n{-}1)(n{-}2)\cdots 1$	$(n{-}2, n{-}1)$	$\tau(\pi)$
$(n{-}1)\boldsymbol{n}(n{-}2)(n{-}3)\cdots 1$	$(n{-}2, n)$	$\sigma(\pi)$
$\boldsymbol{n}(n{-}2)(n{-}3)\cdots 1(n{-}1)$	$(n{-}3, n{-}2)$	$\tau(\pi)$
$(n{-}2)\boldsymbol{n}(n{-}3)(n{-}4)\cdots 1(n{-}1)$	$(n{-}3, n)$	$\sigma(\pi)$
$\cdots$	$\cdots$	$\cdots$
$\boldsymbol{n}1(n{-}1)(n{-}2)\cdots 2$	$(n{-}1, 1)$	$\tau(\pi)$
$1\boldsymbol{n}(n{-}1)(n{-}2)\cdots 2$	$(n{-}1, n)$	$\sigma(\pi)$

Consider the hub seeds defined earlier $\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_{n-1}$. Consider a seed $\boldsymbol{s}_j = s_1 s_2 \cdots s_{n-1}$ with missing symbol x. Observe that $parent(\boldsymbol{s}_j) = \boldsymbol{s}_{j+1}$, where $\boldsymbol{s}_n = \boldsymbol{s}_1$. Let $p_j^1, p_j^2, \ldots, p_j^n$ denote the last n permutations in $Ham(\boldsymbol{s}_j)$ and recall that $p_j^{n-1}, p_j^n = q_j', q_j$ which are in $\mathbf{Q}$. Observe that $parent(\boldsymbol{s}_j)$ is $s_1 x s_2 \cdots s_{n-2}$ and the first n permutations in $Ham(parent(\boldsymbol{s}_j))$ are $q_j', q_j, p_j^1, p_j^2, \ldots, p_j^{n-2}$. From Lemma 3.2 and the definition of $next_{\mathbf{S}}$, the only permutation in $Ham(\boldsymbol{s}_j)$ that has a different successor in $\mathcal{G}_n[\mathbf{S}]$ is p_{n-2}^j; it changes from σ to τ. Thus, starting from the $n+1$st permutation of $Ham(\boldsymbol{s}_1)$, namely $\tau(p_{n-1}^{n-2})$, the successor rule $next_{\mathbf{S}}$ follows the edges in $Ham(\boldsymbol{s}_1)$ until p_1^{n-2}. As just mentioned $next_{\mathbf{S}}(p_1^{n-2}) = \tau(p_1^{n-2})$, which is the $n{+}1$st permutation of $Ham(\boldsymbol{s}_2)$. This process, illustrated below, repeats until reaching p_{n-1}^{n-2} whose successor is the staring permutation $\tau(p_{n-1}^{n-2})$.

q_{n-1}',	q_{n-1},	p_{n-1}^1,	p_{n-1}^2,	$\cdots$	p_{n-1}^{n-2},	$\tau(p_{n-1}^{n-2})$,	$\cdots$	p_1^1,	p_1^2,	$\cdots$	p_1^{n-2},	q_1',	q_1
q_1',	q_1,	p_1^1,	p_1^2,	$\cdots$	p_1^{n-2},	$\tau(p_1^{n-2})$,	$\cdots$	p_2^1,	p_2^2,	$\cdots$	p_2^{n-2},	q_2',	q_2
q_2',	q_2,	p_2^1,	p_2^2,	$\cdots$	p_2^{n-2},	$\tau(p_2^{n-2})$,	$\cdots$	p_3^1,	p_3^2,	$\cdots$	p_3^{n-2},	q_3',	q_3
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
q_{n-2}',	q_{n-2},	p_{n-2}^1,	p_{n-2}^2,	$\cdots$	p_{n-2}^{n-2},	$\tau(p_{n-2}^{n-2})$,	$\cdots$	p_{n-1}^1,	p_{n-1}^2,	$\cdots$	p_{n-1}^{n-2},	q_{n-1}',	q_{n-1}

The rows above correspond to $Ham(\boldsymbol{s}_1), Ham(\boldsymbol{s}_2), Ham(\boldsymbol{s}_3), \ldots, Ham(\boldsymbol{s}_{n-1})$. Concatenating the rows in the middle section yields the Hamilton cycle in $\mathcal{G}_n[\mathbf{S} - \mathbf{Q}]$ constructed by $next_{\mathbf{S}}$. □

For the second step from our outline, the following lemma demonstrates how we can inductively grow the cycle $\mathbf{S} - \mathbf{Q}$ from the previous lemma. For this lemma, let $m = (n{-}1)(n{-}3)!$.

Lemma 3.4 *Let $\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_m$ be an an ordering of all seeds in increasing order by level. Let $\mathbf{S} = \mathbf{perms}(\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_j)$ for some $n{-}1 \le j \le m$. Then $next_{\mathbf{S}}$ partitions $\mathcal{G}_n[\mathbf{S}]$ into two cycles, one for $\mathbf{Q}$ and one for $\mathbf{S} - \mathbf{Q}$.*

Proof. The proof is by induction on j. The base case when $j = n{-}1$ is covered by Lemma 3.3 since the first $n-1$ seeds are the hub seeds with level 0. Consider $\mathbf{S} = \mathbf{perms}(\{\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_j\})$ for $n-1 \le j < m$. Inductively, assume that $next_{\mathbf{S}}$ partitions $\mathcal{G}_n[\mathbf{S}]$ into two cycles, one for $\mathbf{Q}$ and one for $\mathbf{S} - \mathbf{Q}$. Let $\mathbf{S}' = \mathbf{perms}(\boldsymbol{s}_{j+1})$. As mentioned earlier $next_{\mathbf{S}'} = f$ and thus repeated application of $next_{\mathbf{S}'}$ constructs a Hamilton cycle in $\mathcal{G}_n[\mathbf{S}']$. By the ordering of the seeds, $\boldsymbol{s}_{j+1} = s_1 s_2 \cdots s_{n-1}$ has level $\ell > 0$ and all seeds at a smaller level are in $\{\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_j\}$. Thus, by Lemma 2.3 and Lemma 2.4 there is exactly one seed $\boldsymbol{s}$ in $\{\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_j\}$, namely $parent(\boldsymbol{s}_{j+1})$, such that $\mathbf{flower}(\boldsymbol{s}_{j+1}) \cap \mathbf{flower}(\boldsymbol{s})$ is not empty . Moreover this intersection contains the single cyclic permutation $\pi = s_1 x s_2 \cdots s_{n-1}$. Thus, from Lemma 3.2 and using the definition of $Ham(\boldsymbol{s}_j)$, π_1 is the only permutation π' in $\mathbf{S}$ such that $next_{\mathbf{S}\cup\mathbf{S}'}(\pi')$ is not in $\mathbf{S}$. Also, no rotation of π is in $\mathbf{Q}$ by the definition of $\mathbf{Q}$. Thus by replacing the edge $(\pi_1, \sigma(\pi_1))$ in the Hamilton cycle for $\mathcal{G}_n[\mathbf{S} - \mathbf{Q}]$ constructed by $next_{\mathbf{S}}$, with the sub-path of $Ham(\boldsymbol{s}_{j+1})$ starting with π_1 and ending with $\sigma(\pi_1)$, we obtain a Hamilton cycle in $\mathcal{G}_n[\mathbf{S} \cup \mathbf{S}' - \mathbf{Q}]$ constructed by $next_{\mathbf{S}\cup\mathbf{S}'}$. The cycle for $\mathbf{Q}$ remains unchanged. □

From Lemma 2.1, if $\mathbf{S}$ is the set of all seeds, then $\mathbf{perms}(\mathbf{S}) = \mathbf{P}$. From Lemma 3.4, the successor rule $next$ partitions $\mathbf{P}$ into two disjoint cycles: one for the permutations in $\mathbf{Q}$, and one for $\mathbf{P} - \mathbf{Q}$. In the cycle for $\mathbf{Q}$, the permutation $\pi = (n{-}1)(n)(n{-}2)(n{-}3)\cdots 1$ follows $n(n{-}1)\cdots 1$ via a τ operation. By changing this operation to σ in $next$ we obtain a permutation outside $\mathbf{Q}$. Applying this change we obtain the following successor rule.

Hamilton path successor rule for $\mathcal{G}_n$.

Let $\pi = p_1 p_2 \cdots p_n$ be a permutation and let r be the symbol to the right of n when π is considered cyclicly and skipping over p_2. Define the successor rule $next'$ on $\mathcal{G}_n$ as follows:

$$next' = \begin{cases} \tau(\pi) & \text{if } (r, p_2) \in \{(1,2), (2,3), \ldots, (n{-}2, n{-}1), (n{-}1, 1)\} \text{ and } \pi \neq n(n{-}1)\cdots 1; \\ \sigma(\pi) & \text{otherwise.} \end{cases}$$

Starting from the permutation $\pi = (n{-}1)(n)(n{-}2)(n{-}3)\cdots 1$ and applying the successor rule $next$ $n! - 1$ times, we obtain a Hamilton path in $\mathcal{G}_n$. Thus, we obtain the following theorem.

Theorem 3.5 *The successor rule $next'$ can be used to construct a Hamilton path in $\mathcal{G}_n$, for $n > 3$, starting from $\pi = (n{-}1)n(n-2)(n-3)\cdots 1$.*

The Hamilton path in $\mathcal{G}_5$ constructed using $next'$ starting from 45321 is illustrated in Figure 4. A complete C implementation that applies $next'$ to construct a Hamilton path in $\mathcal{G}_n$ is provided in the Appendix.

4 Insights into the Hamilton Cycle Problem

By making a few relatively small changes to the 2-cycle successor rule $next$ we obtain the following successor rule that we claim can be used to construct a Hamilton cycle in $\mathcal{G}_n$ for odd n.

Hamilton cycle successor rule for $\mathcal{G}_n$ for odd n

Let $\pi = p_1 p_2 \cdots p_n$ be a permutation and let r be the symbol to the right of n when π is considered cyclicly and skipping over p_2. Let $\mathbf{S}$ be the set of all rotations of $12\cdots n{-}1$. Define the successor rule $next''$ on $\mathcal{G}_n$ as follows:

$$next'' = \begin{cases} \tau(\pi) & \text{if } (r, p_2) \in \{(1,2), (2,3), \ldots, (n{-}2, n{-}1), (n{-}1, 2)\} \text{ or } p_1 p_3 p_4 \cdots p_n \in \mathbf{S}; \\ \sigma(\pi) & \text{otherwise.} \end{cases}$$

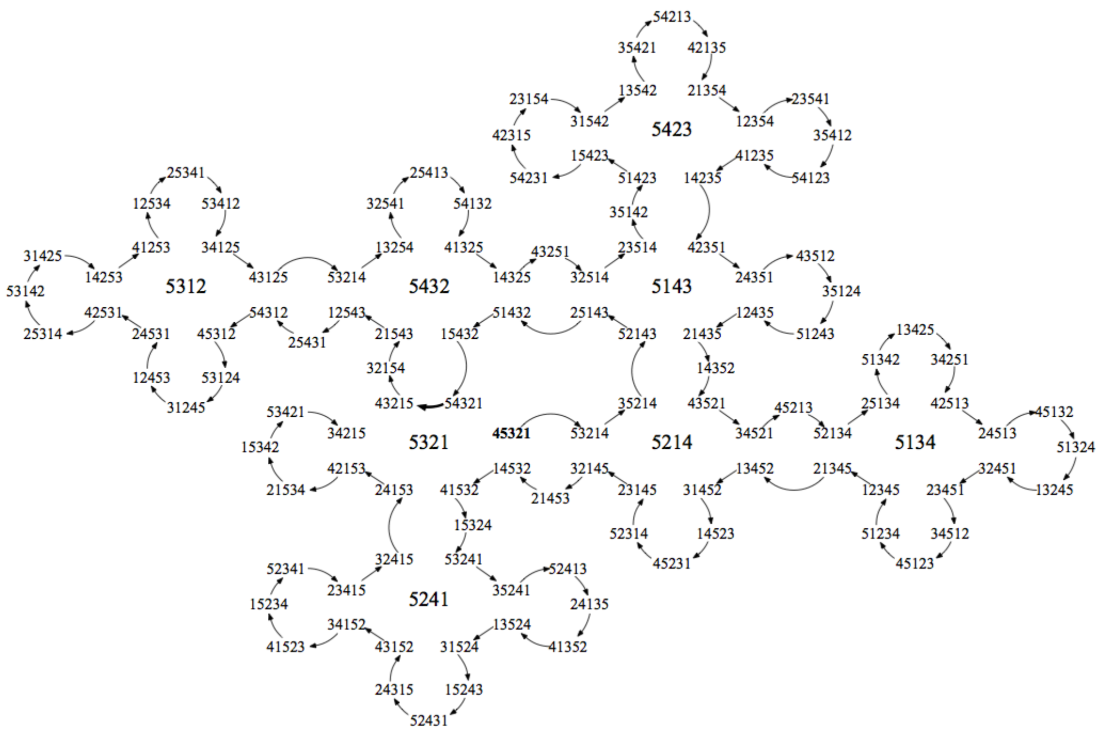


Figure 4: The Hamilton path constructed by $next'$ in the graph $\mathcal{G}_5$ starting from 45321 and ending with 34215.

This rule is based on $(n-1)(n-3)! + 1$ seeds instead of $(n-1)(n-3)!$ seeds in the 2-cycle successor rule. In particular, the condition $(r, p_2) = (n-1, 1)$ is changed to $(r, p_2) = (n-1, 2)$ and this slightly modifies $(n-3)!$ of the seeds and the resulting hub. The other change is the addition of the seed $123 \cdots n-1$. This creates a wheel structure with the additional seed as the center. The resulting directed cycle in the Sigma-Tau graph is traced out using rotation systems in [10]. Our future work is to simplify the proof in [10] using the approach taken in this paper.

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Appendix - C code for Hamilton path in $\mathcal{G}_n$

```
#include <stdio.h>
int n, pi[100];
//--------------------------------------------------------
void Print() {
    for (int i=1; i<=n; i++) printf("%d", pi[i]); printf("\n");
}
//--------------------------------------------------------
void Sigma() {
    int tmp, i;

    tmp = pi[1];
    for (i=1; i < n; i++) pi[i] = pi[i+1];
    pi[n] = tmp;
}
//--------------------------------------------------------
void Tau() {
    int tmp = pi[1];  pi[1] = pi[2];  pi[2] = tmp;
}
//--------------------------------------------------------
int SpecialP() {  // RETURN TRUE IF pi[1..n] = n(n-1)...1

    for (int i=1; i<=n; i++) if (pi[i] != n-i+1) return 0;
    return 1;
}
//--------------------------------------------------------
void Next() {
    int r,i=1;

    while(pi[i] != n) i++;
    if (i == 1) r = pi[3];
    else if (i == n) r = pi[1];
    else r = pi[i+1];

    if (((r < n-1 && pi[2]==r+1) || (r==n-1 && pi[2]==1)) && !SpecialP()) Tau();
    else Sigma();
}
//--------------------------------------------------------
int main() {
    int total=0, TOTAL=1, i;

    printf("Enter n: "); scanf("%d", &n);
    for (i=2; i<=n; i++) TOTAL = TOTAL *i;  // TOTAL = n!

    // INITIAL PERM pi[1..n] = (n-1)n(n-2)(n-3)...1
    pi[1] = n-1;  pi[2] = n;
    for (i=3; i<=n; i++) pi[i] = n-i+1;

    while (total < TOTAL) {
        Print();
        Next();
        total++;
}   }
```