Gamma distribution

The extremely general (and thus not very useful) Gamma distribution has **two parameters**: β called *shape* and θ called *scale*.

The probability density function is:

$$f(x) = \frac{\beta\theta}{\Gamma(\beta)}(\beta\theta x)^{\beta-1}e^{-\beta\theta x} \quad x > 0$$

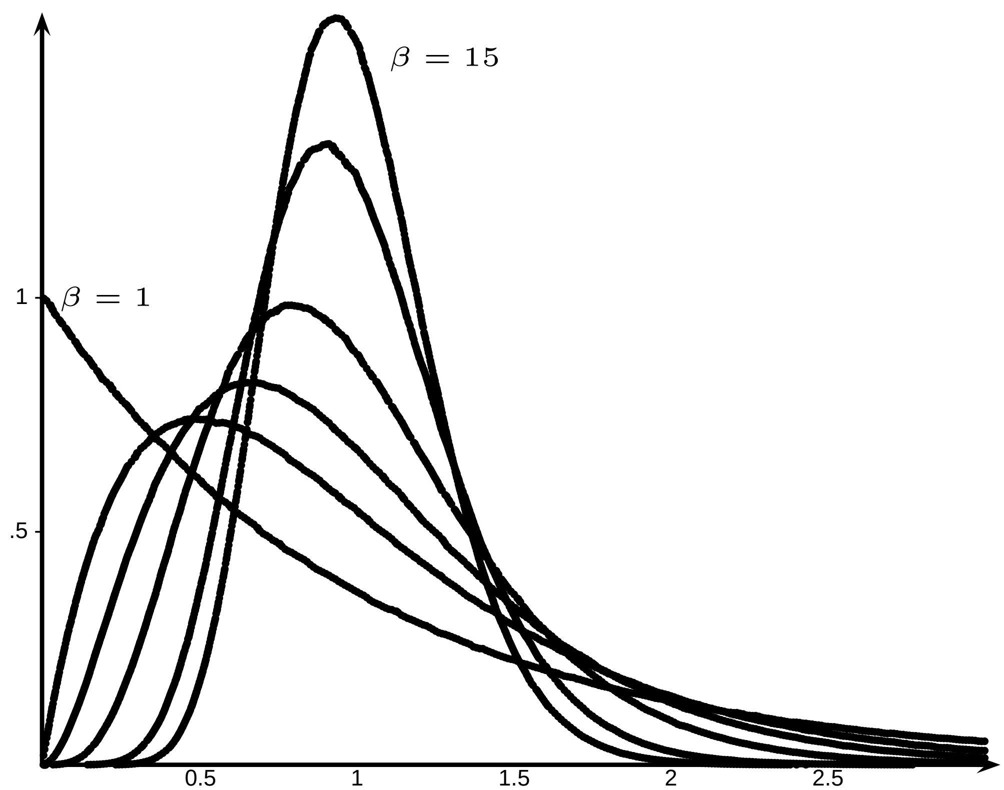
$$\mu = \frac{1}{\theta}$$

$$\sigma^2 = \frac{1}{\beta\theta^2}$$

When $\beta = 1$, the Gamma distribution becomes an exponential distribution with mean $\frac{1}{\theta}$. (θ plays the same role as λ in the Poisson process).

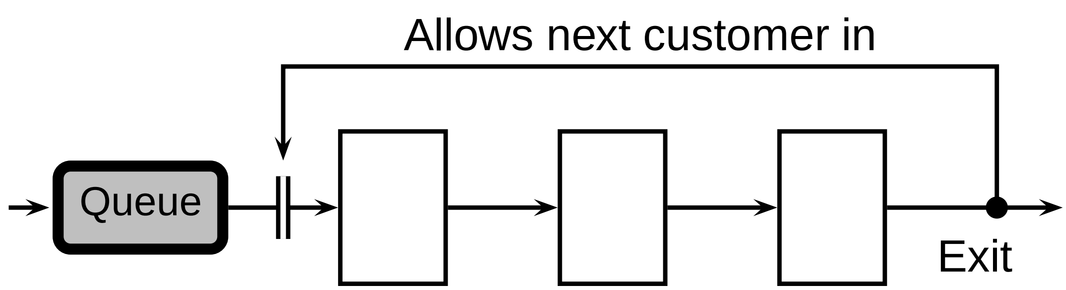
Erlang distribution

For any integer value of β, Gamma turns into Erlang distribution.



Erlang with $\theta = 1$ and $\beta = 1, 2, 3, 5, 10, 15$.

The Erlang distribution models the behaviour of a sequence of β servers put in a pipeline, each server having an exponentially distributed service time. The model additionally requires that only one customer be present in the system pipeline:



An Erlang queue with 3 servers.

Erlang distribution was devised to model the setup of long–distance telephone calls. Nowadays it is widely used to model the behaviour of the process of establishing a "session" (ISO layer 5) or "circuit" in wide–area computer networks.

Generating Erlang variates

It is not easy to generate Gamma variates in the general case. However, when β is integer (i.e. when Gamma becomes Erlang), it is easy:

$$Erlang(\beta, \theta) = \sum_{i=1}^{\beta} Exp(\frac{1}{\beta\theta})$$

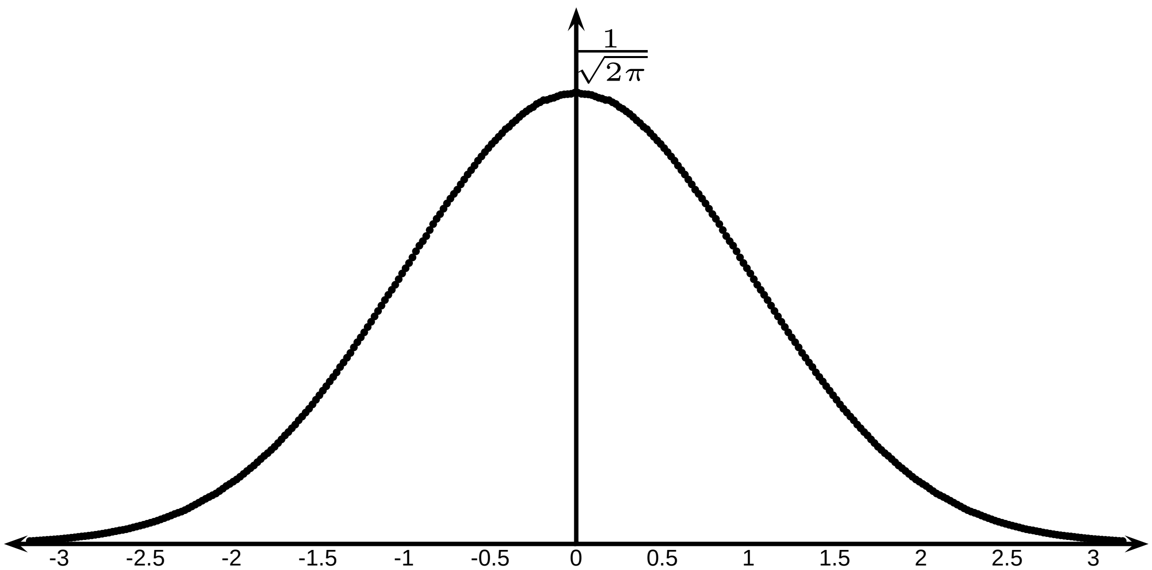
This is not surprising, considering the strong connection between Erlang and a series of exponential servers.

Standard Normal distribution

A **standard Normal distribution** has $\mu = 0, \sigma = 1$. Its pdf is:

$$\phi(\zeta) = \frac{1}{\sqrt{2\pi}} e^{-\zeta^2/2}, \quad -\infty < \zeta < \infty$$

ϕ is symmetric around 0: $\phi(-\zeta) = \phi(\zeta)$ for all ζ.



From ϕ we get the cdf: $\Phi(\zeta) = \int_{-\infty}^{\zeta} \phi(t)dt$

Φ is available in tabulated form.

Non–standard Normal

When we have a normal distribution with a mean of μ and standard deviation σ we can use the transformation:

$$F(x) = \Phi(\frac{x-\mu}{\sigma})$$

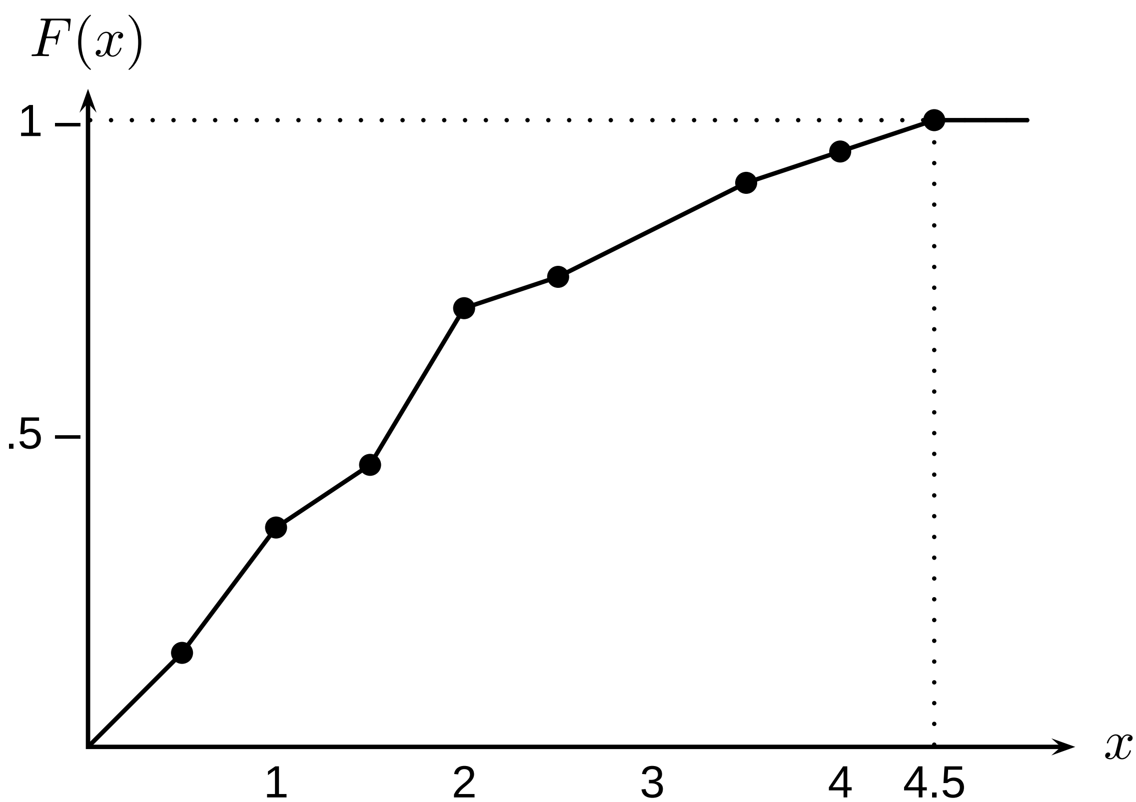
This allows to shift the centre and the shape of the **Bell curve** of the Normal distribution.

Empirical distributions

Suppose that we collected data about 100 cases of repairing a particular type of machine and we know that its repair takes less than 0.5 hour in 15% of cases, etc., as shown:

Time interval	Number of cases	Cumulative frequency
$0.0 < t \leq 0.5$	15	0.15
$0.5 < t \leq 1.0$	20	0.35
$1.0 < t \leq 1.5$	10	0.45
$1.5 < t \leq 2.0$	25	0.70
$2.0 < t \leq 2.5$	5	0.75
$2.5 < t \leq 3.5$	15	0.9
$3.5 < t \leq 4.0$	5	0.95
$4.0 < t \leq 4.5$	5	1.00

We can plot the data points and connect them by straight lines (aka “linear approximation”). If someone wants to be fancy, a nonlinear approximation could be used.



Generating variates

Variates can be generated from the table or from the plot. In either case, we start by generating a $U(0,1)$ variate; let it be R (in calculations, we will take $R = 0.71$).

The method is shown for continuous distributions, but it can be applied to discrete distributions without any change (with no need to interpolate when table lookup is used).

From the table

1. Using the cumulative frequency, we find the interval in which R lies (for 0.71, it is the 2.0–2.5 interval).

2. We interpolate within the interval. Assuming the interval is $t_l, t_h)$, we calculate:

$$\delta = (t_h - t_l) \times \frac{R - F(t_l)}{F(t_h) - F(t_l)}$$

   In the example $R = 0.71$:

$$\delta = (2.5 - 2.0) \times \frac{0.71 - 0.70}{0.75 - 0.70} = 0.1$$

3. The resulting variate is equal to $t_l + \delta$. In the case of the example, the variate is 0.21.

From the plot