A New Way to Represent the Relative Position between Areal Objects

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Abstract—The fuzzy qualitative evaluation of directional spatial relationships (such as "to the right of," "to the south of...,") between areal objects often relies on the computation of a histogram of angles, which is considered to provide a good representation of the relative position of an object with regard to another. In this paper, the notion of the histogram of forces is introduced. It generalizes and may supersede the histogram of angles. The objects (2D entities) are handled as longitudinal sections (1D entities), not as points (0D entities). It is thus possible to fully benefit from the power of integral calculus and, so, ensure rapid processing of raster data, as well as of vector data, explicitly considering both angular and metric information.

Index Terms—Pattern recognition, parameter extraction, spatial relationships, fuzzy subsets.

1 Introduction

Knowing how to apprehend the spatial organization of 2D objects is essential to computer vision (for pattern recognition, image understanding, scene description in natural language, and so forth).

Freeman [5] proposed that the relative position of two objects be described in terms of spatial relationships. He also proposed that the fuzzy set theory be applied because "all-or-nothing" standard mathematical relations are clearly not suited to models of spatial relationships. His ideas were widely adopted. For instance, Gapp [6] used a fuzzy set (the "region of applicability") to define a spatial relation to a reference object; Keller and Sztandera [7] dealt with spatial relationships between fuzzy objects. But, all these authors assimilated 2D crisp objects to very elementary entities such as a point (barycenter) or a (bounding) rectangle. This extremely practical process has often been used, notably for spatial reasoning and representation and processing of qualitative spatial knowledge [4], [12], [13], [20], [22], [23]. However, it is obvious that the procedure cannot be hoped to give a satisfactory modeling of the relationships.

Miyajima and Ralescu illustrated this clearly in [16]. By introducing the notion of the histogram of angles, they also developed the idea that the relative position between two objects can have a representation of its own and can thus be described in terms other than spatial relationships. Numerous studies are based on this notion of histograms of angles [8], [9], [11], [16]. But, the computation of the histogram is costly and only raster data can be considered. Moreover, metric information is not *explicitly* taken into account and, as will be shown here, this is a real handicap.

In this paper, an original notion is presented, that of the *histogram of forces* (also called *F-histogram*). The 2D objects are handled as longitudinal sections, not as points (Fig. 1). It is thus possible to fully benefit from the power of integral calculus and, so, ensure rapid processing of raster data as well as of vector data, explicitly considering both angular and metric information.

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An F-histogram will represent the position of one areal object relative to another. An ideal representation, once computed, is expected to allow rapid fuzzy qualitative evaluation of any spatial relationship. Other applications can also be envisaged (e.g., [24]). At the very least, an F-histogram will have to allow assessment of the directional relationships (such as "`above`," "`to the right of`," etc.). In Section 2, we specify the type of object we want to handle and we set axiomatic properties to the fuzzy directional spatial relations between these objects. Section 3 looks at the search for functions handling longitudinal sections. This search is guided by the previous axioms. In Section 4, we study how an F-histogram can be produced from the datum of such a function and, then, show how the notion of the histogram of forces enables the definition of spatial relations that indeed satisfy the axiomatic properties. Finally, a comparative study is proposed in Section 5 and our conclusion appears in Section 6.

2 Directional Spatial Relations

The Euclidean affine plane is referred to as a directional orthogonal frame $(O, \vec{i}, \vec{j})$. Let θ and v be two reals, $\vec{i_\theta}$ and $\vec{j_\theta}$ be the respective images of $\vec{i}$ and $\vec{j}$ through a θ-angle rotation, and $\Delta_\theta(v)$ the oriented line whose frame is defined by the vector $\vec{i_\theta}$ and the point of coordinates $(0, v)$—relative to $(O, \vec{i_\theta}, \vec{j_\theta})$—(Fig. 2).

2.1 Relations between Points

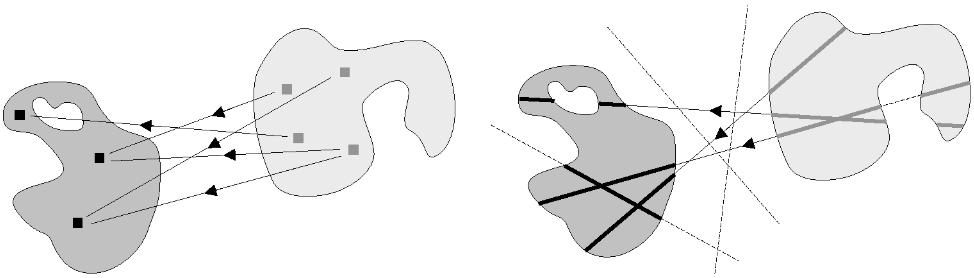
There is no controversy about directional spatial relations between points. We believe that all authors will accept the following formulations. A *fuzzy directional spatial relation between points* is a fuzzy binary relation $\mathcal{R}_\alpha$ between points, where α represents any real. $\mathcal{R}_\alpha$ connects any couple (A, B) of distinct points with an element of interval [0, 1]. This element, $\mathcal{R}_\alpha(A, B)$, is put in the same category as the degree of truth of the proposition $A\mathcal{R}_\alpha B$ stated as "`A is in direction` α `of B`." Point A is the *argument* of the proposition and point B is the *referent*. For some values of α, according to the context, particular conventions are generally used. For example, the relation $\mathcal{R}_0$ is sometimes expressed as "`to the right of`," "`to the east of ...`," and the $A\mathcal{R}_0 B$ proposition stated as "`A is to the right of B`," "`A is to the east of B`...." The directional spatial relations between points are defined from a fuzzy subset of $\mathbb{R}$. Its membership function μ is continuous, with period 2π, even, decreasing on $[0, \pi]$, and takes the value 1 at 0 and the value 0 at $\pi/2$. Let α and β be two reals and A and B be distinct points. If β is an $(\vec{i}, \vec{BA})$ angle measure, then (Fig. 3): $\mathcal{R}_\alpha(A, B) = \mu(\beta - \alpha)$.

2.2 Relations between Areal Objects

As underlined by Rosenfeld and Kak [21], although it is easy to define spatial relations between points, the problem gets complex when 2D objects are processed and when parameters such as shape, orientation, and dimensions are involved. The type of the objects we are going to handle is specified as follows:

Definition 1. *A* crisp object E *is a nonempty bounded set of points, equal to its interior closure and such that, for any real* θ *and for any real* v, $E \cap \Delta_\theta(v)$ *is the union of a finite number of mutually disjoint segments. The set* $E \cap \Delta_\theta(v)$, *denoted* $E_\theta(v)$, *is a* longitudinal section *of* E *(Fig. 2).*

The topologic condition ($\bar{\mathring{E}} = E$) enables us to consider regular closed sets only [26], which do not include any "grafting" such as an arc or isolated point. The adjective "crisp" will henceforth often be omitted (the definition of a "fuzzy" object is given in Appendix B). Let us now examine a family of fuzzy binary relations between objects, corresponding to a family of directional spatial relations. (They are, therefore, assumed to be indifferent to the context: The directional position of an object with regard to another is in no way



(a) (b)

Fig. 1. The assessment of relative positions. (a) The classical approach: The objects are handled as points. (b) The proposed approach: The objects are handled as longitudinal sections.

affected by the rest of the scene.) Terminology and notations are as in Section 2.1. Given a couple of objects (A, B), it is legitimate to assume that if any of the considered relations is defined at (A, B), then *all* are. Such a couple will be termed *assessable*. It is also legitimate to assume the following properties:

[A1] Let A and B be two crisp objects and α and β be two reals. Denote by $t_{\vec{u}}$ the translation of vector $\vec{u}$. There exists a real number k_0 such that, for any k greater than k_0, the $(t_{k\cdot\vec{i}_\beta}(A), B)$ couple is assessable. Moreover,

$$\lim_{k\to+\infty} \mathcal{R}_\alpha(t_{k\cdot\vec{i}_\beta}(A), B) = \mu(\beta - \alpha)$$

[A2] Let A and B be two crisp objects and α a real. If (A, B) is assessable, then (B, A) is also assessable and

$$\mathcal{R}_{\alpha+\pi}(B, A) = \mathcal{R}_\alpha(A, B)$$

[A3] Let A and B be two crisp objects, α a real and sym a $\Delta_\beta(v)$-axis orthogonal symmetry. If (A, B) is assessable, then $(sym(A), sym(B))$ also is and

$$\mathcal{R}_{2\beta-\alpha}(sym(A), sym(B)) = \mathcal{R}_\alpha(A, B)$$

[A4] Let A and B be two crisp objects, α a real, and dil a dilatation with a strictly positive ratio. If (A, B) is assessable, then $(dil(A), dil(B))$ also is and $\mathcal{R}_\alpha(dil(A), dil(B)) = \mathcal{R}_\alpha(A, B)$

We term these the *basic axiomatic properties* (Fig. 4). [A1] signifies that two objects can be assimilated to points if they are distant enough. [A3] and [A4] define the behavior of the directional spatial relations towards similitudes. [A4] means that the relations are not sensitive to scale changes, [A3] that neither a space dimension nor a direction are preferred. [A2] brings out the notion of *semantic inverse* (according to Freeman [5]). A is thus to the left of B as B is to the right of A. As can be observed through the literature (especially in the papers by Freeman [5] and Retz-Schmidt [18]), the points of view stated above are globally set aside by works in the fields of linguistics and psychology. Nevertheless, they are, in a more or less explicit way, widely adopted by computer scientists [7], [10], [11], [16], [17]. However, we must accept our limited understanding of the mechanisms of human perception.

3 Handling of Longitudinal Sections

3.1 Introduction

Let T be the set of triples $(\theta, A_\theta(v), B_\theta(v))$, where θ and v describe $\mathbb{R}$ and A and B describe the set of crisp objects of the plane. Remember that $A_\theta(v)$ is a longitudinal section of A (see Section 2.2, Definition 1), $A_\theta(v) = A \cap \Delta_\theta(v)$. Likewise, $B_\theta(v) = B \cap \Delta_\theta(v)$. How can we define a function F from T into $\mathbb{R}_+$ such that, for any couple (A, B) of objects, the set

$$\{(\Delta_\theta(v), F(\theta, A_\theta(v), B_\theta(v)))\}_{(\theta,v)\in\mathbb{R}^2}$$

of data can represent the relative position of A with regard to B? To answer this question, we treat $F(\theta, A_\theta(v), B_\theta(v))$ as a weight. The weight of the arguments which an observer *whose scope of vision is limited to* $\Delta_\theta(v)$ will, nevertheless, find to support the proposition "A `is in direction` θ `of` B" (Fig. 5). Our thought process led us from the directional spatial relations to F—for the handling of longitudinal sections, then from F to a second function f—for a thinner handling of segments and, then, from f to a third function φ—for the handling of points. The axiomatic properties [A1] to [A4] actually induce axiomatic properties on F. The existence of f is deduced from these properties. Axiomatic properties are also induced on f and the existence of φ is finally deduced from the properties of f. This is a natural progression because a longitudinal section is a union of segments and a segment is a union of points.

The functions φ, f, and F are presented in Section 3.2. In order to be more concise and to increase readability, we have decided to introduce first φ, then f, then F, and not the opposite. Concrete examples of functions φ and f are given in Section 3.3. Within the article, numerous properties are referred to and some propositions are formulated. The properties are written down in Appendix A and their principal links illustrated by Fig. 15. For each proposi-

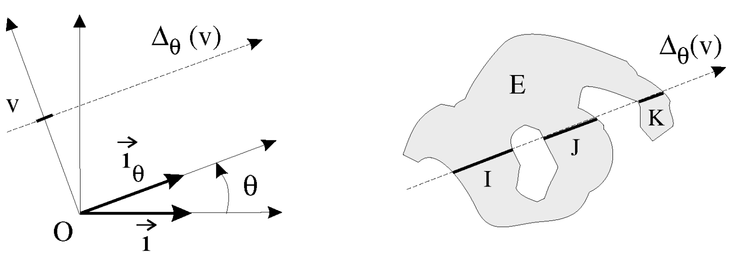


Fig. 2. Oriented straight lines and longitudinal sections. $E_\theta(v) = E \cap \Delta_\theta(v) = I \cup J \cup K$.

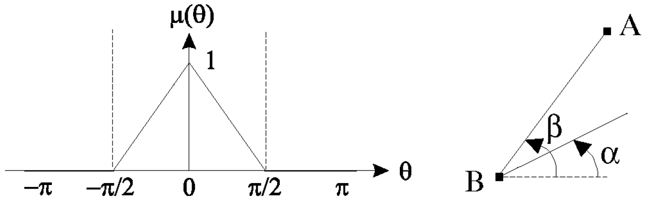


Fig. 3. Example of directional spatial relations between points. The degree of truth of the proposition "`A is in direction` α `of B`" is $\mu(\beta - \alpha)$.

tion, the proof is given and the reciprocal studied in [14]. The French-speaking reader is also invited to consult [15].

3.2 General Presentation of Functions φ, f, F

3.2.1 Handling of Points: the Function φ

The handling of couples of points is carried out by way of a map φ from $\mathbb{R}$ into $\mathbb{R}_+$, null on $\mathbb{R}_-$, continuous on $\mathbb{R}_+^*$. Let A and B be two objects, θ and v two reals, M a point of $A_\theta(v)$, and N a point of $B_\theta(v)$. We consider the (M, N) couple an argument put forward to support the proposition "A `is in direction` θ `of` B" (Fig. 5). The value $\varphi(x_M - x_N)$, where x_M and x_N refer to the respective abscissas of M and N on $\Delta_\theta(v)$, is the weight of this argument. It does not depend on θ but only on the relative position of M and N on $\Delta_\theta(v)$. This is in accordance with [A3]. If x_M is lower than or equal to x_N, i.e., if $x_M - x_N$ belongs to $\mathbb{R}_-$, the advanced argument is not acceptable and, therefore, its weight $\varphi(x_M - x_N)$ is null. Continuity is an analytical property which is often assessed as a minimum requirement because it is verified by most physical phenomena (at least on the human scale). It seems natural to accept it here. However, continuity can obviously not be required at 0. If $x_M - x_N$ is strictly positive, then M *is* in direction θ of N, and if $x_M - x_N$ is strictly negative, then M *is not* in direction θ of N.

3.2.2 Handling of Segments: the Function f

The handling of couples of segments is carried out by way of a function f from $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ into $\mathbb{R}_+$. Let (I, J) be a couple of aligned segments. There exists an infinite number of θ values (equal modulo π) such that (θ, I, J) belongs to T. Let us pick such a value. There now exists a real number v, and only one, such that the oriented straight line $\Delta_\theta(v)$ includes I and J. Let the coordinates—relative to the frame associated with $\Delta_\theta(v)$—at the ends of segments I and J be noted $a_I^\theta, b_I^\theta, a_J^\theta$, and b_J^θ, with a_I^θ lower than b_I^θ, and a_J^θ lower than b_J^θ (Fig. 6). Both $b_I^\theta - a_I^\theta$ and $b_J^\theta - a_J^\theta$ are positive values not depending on θ, noted d_I and d_J. They correspond to the segment lengths. The difference $a_I^\theta - b_J^\theta$, depending on θ, is noted D_{IJ}^θ.

Let A and B be two objects, θ and v two reals, I one of the segments that form $A_\theta(v)$, and J one of the segments that form $B_\theta(v)$. We consider the (I, J) couple an argument put forward to support the proposition "A `is in direction` θ `of` B." The value $f(d_I, D_{IJ}^\theta, d_J)$ represents the weight of this argument. It only depends on the lengths of I and J and on the relative position of these segments on $\Delta_\theta(v)$. At this stage, it seems natural to estimate $f(d_I, D_{IJ}^\theta, d_J)$ by summing the weights $\varphi(x_M - x_N)$ of the (M, N) arguments, where M and N describe I and J, respectively:

$$\begin{aligned} f(d_I, D_{IJ}^\theta, d_J) &= \int_{a_I^\theta}^{b_I^\theta} \left(\int_{a_J^\theta}^{b_J^\theta} \varphi(u - v).dv \right).du \\ &= \int_{D_{IJ}^\theta + d_J}^{d_I + D_{IJ}^\theta + d_J} \left(\int_0^{d_J} \varphi(u - v).dv \right).du\ . \end{aligned}$$

(Note that addition is not the sole information fusion operator. Other operators can be considered [14].)

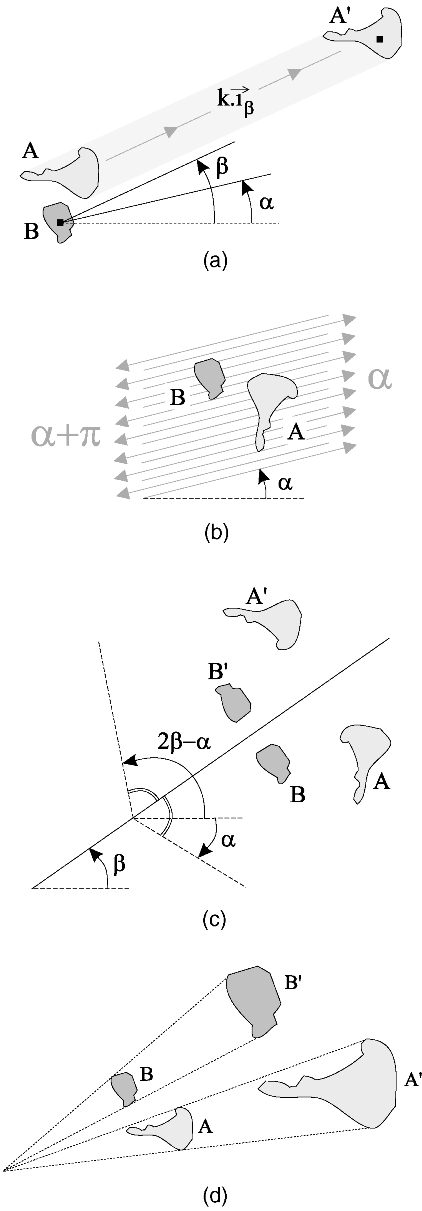


Fig. 4. The basic axiomatic properties. (a) [**A1**] The position of A' with regard to B is approximately like the position of any point of A' with regard to any point of B. (b) [**A2**] B is in direction $\alpha + \pi$ of A as A is in direction α of B. (c) [**A3**] A' is in direction $2\beta - \alpha$ of B' as A is in direction α of B. (d) [**A4**] The position of A' with regard to B' is like the position of A with regard to B.

Definition 2. *Let φ be a map from $\mathbb{R}$ into $\mathbb{R}_+$. The function f from $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ into $\mathbb{R}_+$ defined by the following formula is called the* function generated *by φ:*

$$f(x, y, z) = \int_{y+z}^{x+y+z} \left(\int_0^z \varphi(u - v) dv \right) du.$$

Proposition 1. *Let φ be a map from $\mathbb{R}$ into $\mathbb{R}_+$ and f the function generated by φ. If φ is null on $\mathbb{R}_-$, continuous on $\mathbb{R}_+^*$, and if there exists no nonempty open interval E of $\mathbb{R}_+^*$, such that, φ is null on E, then f satisfies properties [P1] to [P3].*

Note that all the properties are given in Appendix A. When $x + y + z$ is strictly negative ([P1]), the couple (I, J) is not a good argument in favor of the proposition "A `is in direction` θ `of`

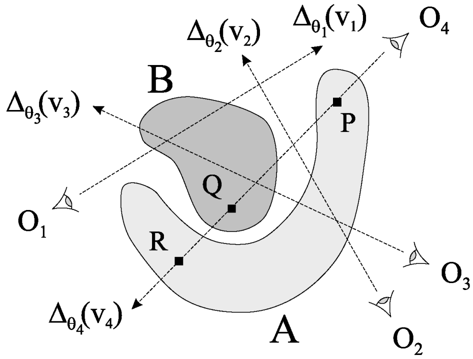


Fig. 5. Handling of longitudinal sections. Interpretation of $\varphi/f/F$. Observer O_4 looks for arguments in favor of the proposition "A is in direction θ_4 of B." (R, Q) is a good argument but not $(P, Q) : \varphi(x_P - x_Q) = 0$ (Section 3.2.1). As for the other observers, they cannot produce an argument to support their respective propositions.

B" (Fig. 6b). When y is strictly positive ([P2]), the couple makes up a good argument (Fig. 6a). The signification of [P2] is indicated in Fig. 7.

3.2.3 Handling of Longitudinal Sections: the Function F

The handling of couples of longitudinal sections is carried out by way of the function F from T (see Secton 3.1) into $\mathbb{R}_+$. Let A and B be two objects and θ and v two reals. There exists one set $\{I_i\}_{i \in 1..n}$ of mutually disjoint segments, and only one, such that $A_\theta(v) = \cup_{i \in 1..n} I_i$. Likewise, there exists one set $\{J_j\}_{j \in 1..m}$ of segments such that $B_\theta(v) = \cup_{j \in 1..m} J_j$. We consider the $(A_\theta(v), B_\theta(v))$ couple an argument put forward to support the proposition "A is in direction θ of B." The weight of this argument is represented by $F(\theta, A_\theta(v), B_\theta(v))$. It seems natural to estimate it by summing the weights $f(d_I, D^\theta_{IJ}, d_J)$ of the (I, J) arguments, where I and J describe $\{I_i\}_{i \in 1..n}$ and $\{J_j\}_{j \in 1..m}$, respectively:

$$F(\theta, A_\theta(v), B_\theta(v)) = \sum_{i \in 1..n, j \in 1..m} f(d_{I_i}, D^\theta_{I_i J_j}, d_{J_j}) .$$

Definition 3. *Let f be a function from $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ into $\mathbb{R}_+$. The function F from T into $\mathbb{R}_+$ defined by the following formula (where $\{I_i\}_{i \in 1..n}$ denotes the sole set of mutually disjoint segments such that $I = \cup_{i \in 1..n} I_i$, and $\{J_j\}_{j \in 1..m}$ the sole set of segments such that $J = \cup_{j \in 1..m} J_j$), is called the* function generated by *f:*

$$F(\theta, I, J) = \sum_{i \in 1..n, j \in 1..m} f(d_{I_i}, D^\theta_{I_i J_j}, d_{J_j}).$$

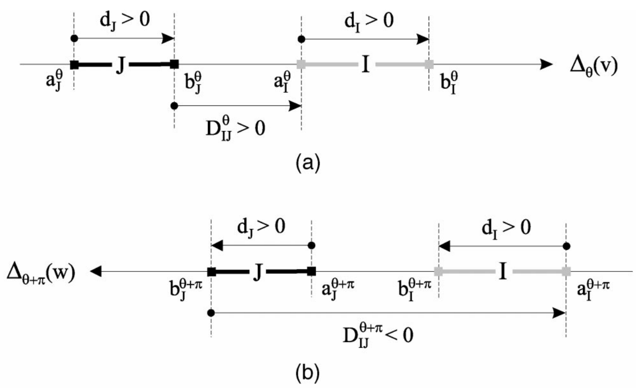


Fig. 6. Longitudinal sections of disjoint convex objects.

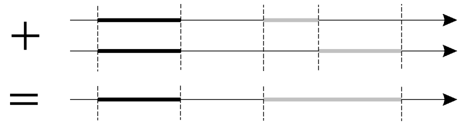


Fig. 7. Property [P2].

Proposition 2. *Let f be a function from $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ into $\mathbb{R}_+$ and let F be the function generated by f. If f satisfies properties [P1] to [P3], then F satisfies properties [P7] to [P10].*

In Section 4, we present how to define a family of directional spatial relations from the datum of a function F handling longitudinal sections. We can already guess that [P9] and [P10] will help guarantee the basic axiomatic properties [A2] and [A3]; the similarities between [P9] and [A2] on the one hand and between [P10] and [A3] on the other hand are obvious. As for [P7] and [P8], even if the link is more difficult to establish, they help guarantee [A1] (see Fig. 15, in Appendix A). What about A4? This point is dealt with in Section 3.3.

3.3 A Particular Family of Functions $\varphi/f/F$

For any real number r, let φ_r be the map from $\mathbb{R}$ into $\mathbb{R}_+$, null on $\mathbb{R}_-$ such that $\forall d \in \mathbb{R}^*_+, \ \varphi_r(d) = 1/d^r$. Let f_r be the function generated by φ_r and F_r the one generated by f_r.

Proposition 3. *Let r be a real number. Function f_r satisfies [P1] to [P3] and also [P4].*

Proposition 4. *Let f be a function from $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ into $\mathbb{R}_+$ and let F be the function generated by f. If f satisfies properties [P1] to [P4], then F satisfies properties [P7] to [P10] and also [P11].*

So, the F_r functions satisfy property [P11]. The importance of the $\varphi_r/f_r/F_r$ families is due to this characteristic, which allows [A4] to be guaranteed (see Fig. 15, in Appendix A). The importance of these families is also due to the particularities of $\varphi_0/f_0/F_0$ and $\varphi_2/f_2/F_2$. Let A and B be two objects, θ and v two reals, M a point of $A_\theta(v)$, and N a point of $B_\theta(v)$. Let us assume that the (M, N) couple is a good argument in favor of the proposition "A is in direction θ of B" (i.e., M is in direction θ of N). Function φ_0 confers to this argument a weight independent of the distance MN. This independence naturally affects f_0, and f_0 satisfies [P5] (Fig. 8a). Now, if M and N are material points of unit mass, then the weight assigned by φ_2 represents (to within a factor of multiplication) the gravitational force exerted by M on N (a force which tends to move N in direction θ). This gives property [P6] to f_2, which is also particularly interesting; f_2 is not sensitive to scale changes (Fig. 8b). We will come back to these particularities in the following sections.

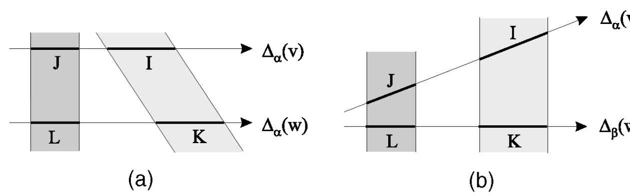
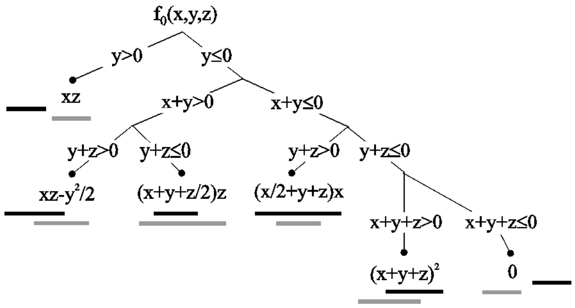
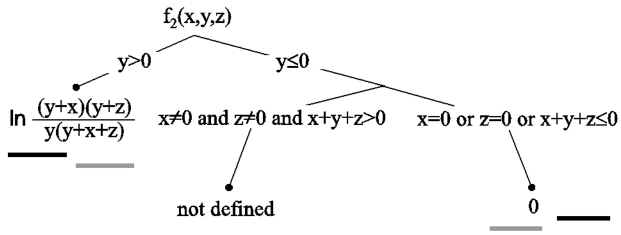


Fig. 8. Properties [P5] and [P6]. (a) Independence from distance: $F(\alpha, I, J) = F(\alpha, K, L)$ ([P5]) (b) Independence from scale: $F(\alpha, I, J) = F(\beta, K, L)$ ([P6]).

Proposition 5. *f_0 satisfies properties [P1] to [P4] and [P5] and is defined by the following tree:*



Proposition 6. *f_2 satisfies properties [P1] to [P4] and [P6] and is defined by the following tree:*



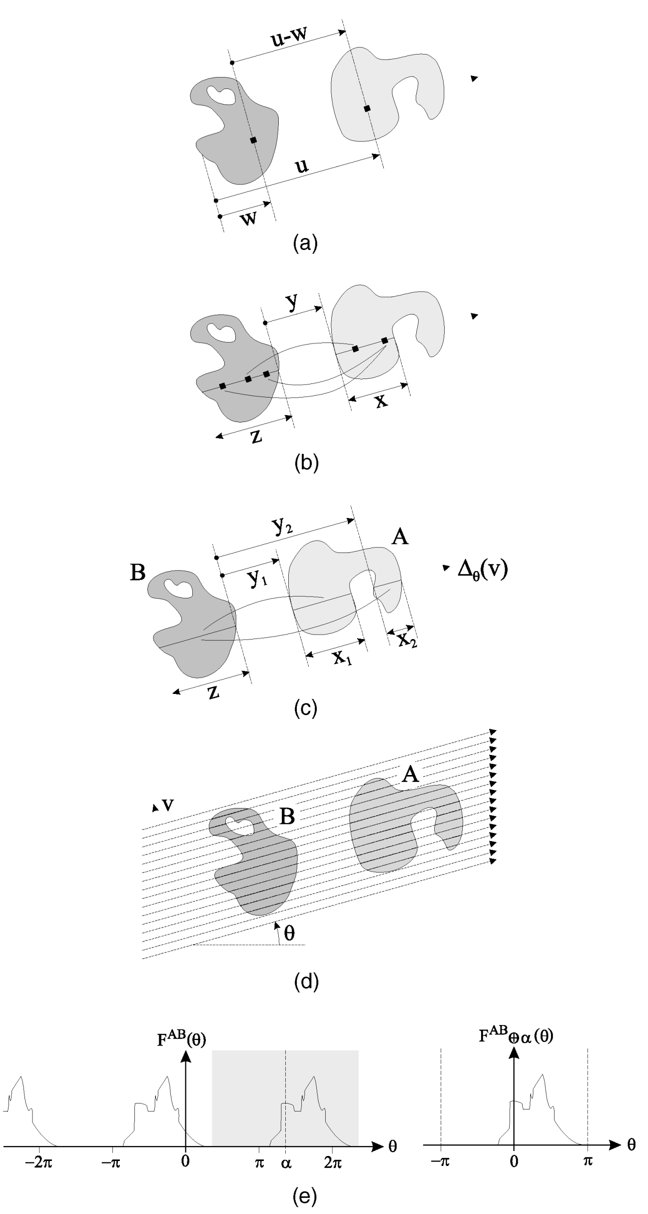


Fig. 9. Computation of F-Histograms and evaluation of directional spatial relations. Recap. (a) Handling of points (Section 3.2.1): $\varphi(u-w)$. (b) Handling of segments (Section 3.2.2): $f(x,y,z)=\int_{y+z}^{x+y+z}\left(\int_0^z \varphi(u-w).dw\right).du$. (c) Handling of longitudinal sections (Section 3.2.3): $F(\theta,A_\theta(v),B_\theta(v))=f(x_1,y_1,z)+f(x_2,y_2,z)$. (d) Handling of directions (Section 4.1): $F^{AB}(\theta)=\int_{-\infty}^{+\infty}F(\theta,A_\theta(v),B_\theta(v)).dv$. (e) Handling of histograms (Section 4.2): $\mathcal{R}_\alpha(A,B)=H(F^{AB}\oplus\alpha)$.

Propositions 5 and 6 give the expressions of f_0 and f_2. Note that for these functions, a couple of aligned segments is seen as a triple (x,y,z) of real numbers (Section 3.2.2); x denotes the length of the referent segment (represented in the trees by a black line) and z the length of the argument segment (represented in gray). The algebraic expression corresponding to the process of a given couple depends on the relative position of the segments.

4 Histograms of Forces and Directional Relations

4.1 F-Histograms

For any function F from T into $\mathbb{R}_+$ and for any couple (A,B) of objects, let us denote by F^{AB} the function defined by:

$$F^{AB}| \qquad \mathbb{R}\rightarrow\mathbb{R}_+$$
$$\theta\longmapsto\int_{-\infty}^{+\infty}F(\theta,A_\theta(v),B_\theta(v)).dv$$

If F^{AB} is defined on $\mathbb{R}$ and if F is defined at $(\theta,A_\theta(v),B_\theta(v))$ for any couple (θ,v) of reals, the couple (A,B) is termed F-*assessable*. $F^{AB}(\theta)$ then represents the total weight of the arguments stated in favor of the proposition "A `is in direction` θ `of` B." In Section 3.3, we noticed that φ_0 (which generates F_0) is independent of distance. The consequence is that F_0^{AB} and $\mathcal{A}^{AB}$—the histogram of angles associated with the digitized objects [16]—are fundamentally equivalent, even though the first arises from the continuous case and the second from the discrete case. For instance, it is shown in [14] that, for the horizontal and vertical directions (i.e., θ belonging to $\{-\frac{\pi}{2},0,\frac{\pi}{2},\pi\}$), the values $F_0^{AB}(\theta)$ and $\mathcal{A}^{AB}(\theta)$ are strictly equal. The equivalence between F_0^{AB} and $\mathcal{A}^{AB}$ is clearly illustrated in Section 5. In Section 3.3, we also noticed that the weights assigned by φ_2 can be interpreted as gravitational forces. Thus, if each object is a homogeneous

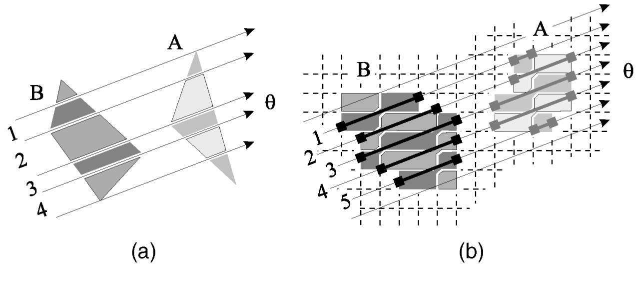


Fig. 10. The evaluation of $F^{AB}(\theta)$ is based on the partitioning of the objects. (a) Case of vector data. Four couples of trapeziums are processed. (b) Case of raster data. Five couples of segments are processed.

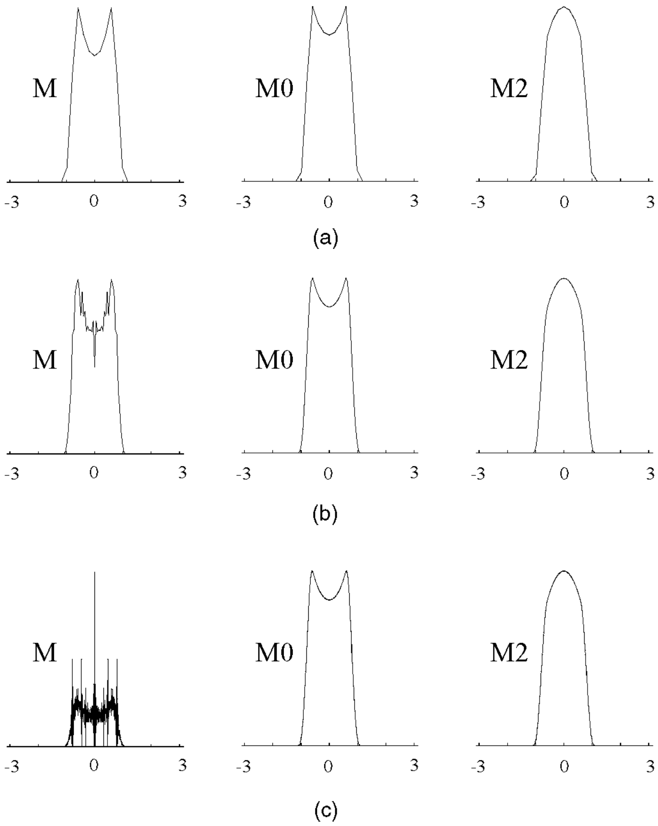


Fig. 11. Histograms of angles (**M**) and histograms of forces (**M0**, **M2**) associated with test image 1. On the x-axes, directions θ are expressed in radians. The y-axes are dimensionless: indeed, the fact that H_μ^M satisfies property [P14] (see Section 4.2) allows the histograms to be normalized. Note that the intrinsic form of a histogram of angles only appears for high values of the digitization step. But, in this case, the curves drawn are angular. (a) Digitization step: ≈ 11.3 degrees ($360/32$ degrees). Computing times: 0.7 CPU second (**M**) and 0.3 CPU second (**M0**, **M2**). (b) Digitization step: ≈ 2.8 degrees ($360/128$ degrees). It is the chosen step (test images 1 to 8). Computing times: 0.7 CPU second (**M**) and 0.9 CPU second (**M0**, **M2**). (c) Digitization step: ≈ 0.3 of a degree ($360/1024$ of a degree). Computing times: 0.7 CPU second (**M**) and 7.7 CPU seconds (**M0**, **M2**).

material surface of unit specific mass, $F_2^{AB}(\theta)$ represents the scalar resultant of elementary forces of gravity; the forces exerted by the A points on those of B, each tending to move B in direction θ.

That is why function F^{AB}, for any F-assessable couple (A, B), is called the *histogram of forces associated with* (A, B) *via* F. We can consider this histogram—also called the *F-histogram associated with* (A, B)—to result from the compression of the

$$\{(\Delta_\theta(v), F(\theta, A_\theta(v), B_\theta(v)))\}_{(\theta,v)\in\mathbb{R}^2}$$

data set: It represents the relative position of object A with regard to object B. Propostion 7 specifies the F_r-assessable couples of objects—i.e., those to which a F_r-histogram can be associated.

Proposition 7. *Let r be a real number. If r belongs to $]-\infty, 1[$, then any couple of objects is F_r-assessable. If r belongs to $[1, 2[$, then the F_r-assessable couples of objects are those with disjoint interiors (disjoint or tangent objects). If r belongs to $[2, +\infty[$, only the couples of disjoint objects are F_r-assessable.*

4.2 Handling of Histograms: the Function H

A histogram of forces is a map from $\mathbb{R}$ into $\mathbb{R}_+$ with period 2π. For any such map h and for any real α, let us denote by $h \oplus \alpha$ the function defined by:

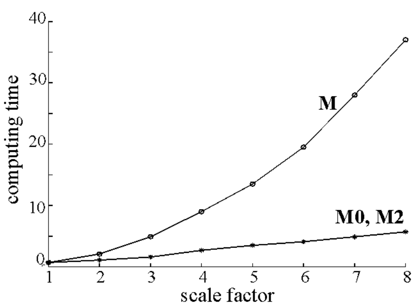


Fig. 12. Scale effect on processing time. The scale factor corresponds to the area of the processed image (test image 1). Unit area 160×80 pixels. Unit time is the CPU second.

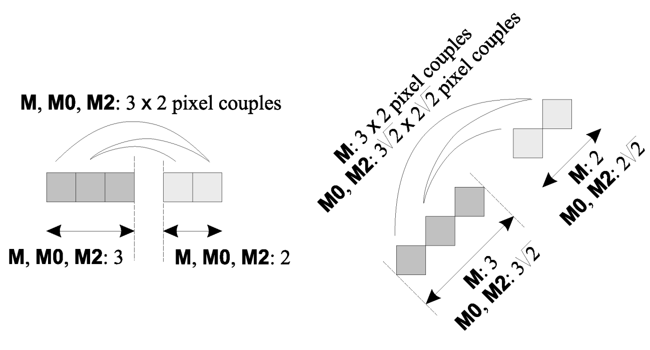
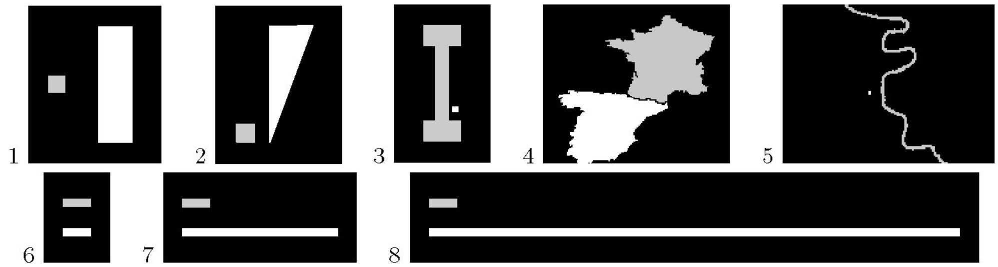


Fig. 13. Isotropy (**M0, M2**), and anisotropy (**M**).



	1			2			3			4			5			6			7			8		
	M	M0	M2	M	M0	M2	M	M0	M2	M	M0	M2	M	M0	M2	M	M0	M2	M	M0	M2	M	M0	M2
Right	**77**	**77**	**82**	33	41	**64**	26	38	**58**	09	13	19	03	03	02	11	11	10	**78**	**57**	31	**94**	**71**	32
Left	00	00	00	00	00	00	02	05	07	48	47	40	41	51	**75**	11	11	10	11	11	10	11	11	10
Above	24	21	18	**68**	**60**	40	56	56	48	00	00	01	45	37	21	00	00	00	00	00	00	00	00	00
Below	24	21	18	05	05	05	**84**	**67**	33	**53**	**55**	**65**	**71**	**59**	28	**89**	**89**	**91**	29	51	**75**	09	36	**74**

Fig. 14. Test Images and Result Table. Argument A appears in white and referent B in gray. The results are given in hundredths. **Image 2**—**M2** is the only one to affirm that A is more to the right of B, even though it gives a certain credit to the proposition "A is above B." **Image 3** (this configuration has been proposed in [16])—**M** assesses A to be clearly more below B (or even above B!) than to the right of it. **Image 4**—Good pupils say, "Spain is to the south of France." And you will certainly not make them believe that it is to the west. But, when pressed, they may just agree that it does extend toward the west. **M2** shares this point of view much better than the other methods. **Image 5**—**M2** states that the "house" is west of the "river." **M** does not share this opinion. **Images 6, 7, and 8**—As argument A becomes longer, **M** quickly affirms that A is essentially located to the right of referent B. **M0** eventually shares this point of view, but later on and in a less definite way. Note that in **Image 8**, **M** assesses A to be rather to the left of B than below it! **M2** is alone to maintain that A essentially remains below B. When the length of object A tends toward infinity, TO_THE_RIGHT_OF (A, B) increases toward a limit value near $1/3$, whereas BELOW (A, B) decreases towards a value near $3/4$.

$$h \oplus \alpha| \quad [-\pi,\pi] \rightarrow \mathbb{R}_+$$
$$\theta \longmapsto h(\theta+\alpha)\ .$$

It is an element of $Map([-\pi,\pi],\mathbb{R}_+)$—i.e., a map from $[-\pi,\pi]$ into $\mathbb{R}_+$. The handling of histograms of forces – to evaluate the directional spatial relations—is carried out by means of a map H from $Map([-\pi,\pi],\mathbb{R}_+)$ into $[0,1]$.

Definition 4. *Let F be a function from T into $\mathbb{R}_+$ and let H be a map from $Map([-\pi,\pi],\mathbb{R}_+)$ into $[0,1]$. Consider the family $(\mathcal{R}_\alpha)_{\alpha\in\mathbb{R}}$ of fuzzy binary relations defined on exactly the set of F-assessable couples by: $\mathcal{R}_\alpha(A,B) = H(F^{AB}\oplus\alpha)$. We call $(\mathcal{R}_\alpha)_{\alpha\in\mathbb{R}}$* the family of directional spatial relations generated by F and H.

Proposition 8. *Let F be a function from T into $\mathbb{R}_+$, let H be a map from $Map([-\pi,\pi],\mathbb{R}_+)$ into $[0,1]$, and μ the membership function of a fuzzy subset "directional spatial relations between points" (see Section 2.1). Let $(\mathcal{R}_\alpha)_{\alpha\in\mathbb{R}}$ be the family of relations generated by F and H. We have:*

IF F satisfies $[P7]$ to [P11] and H and μ share [P12] and H [P13] [P14]	**THEN** $(\mathcal{R}_\alpha)_{\alpha\in\mathbb{R}}$ and μ share [A1] and $(\mathcal{R}_\alpha)_{\alpha\in\mathbb{R}}$ satisfies [A2] to [A4].

The principal links between [P7] to [P14] and [A1] to [A4] are illustrated by Fig. 15 in Appendix A. Given Proposition 8, families of directional spatial relations (in the sense proposed in Section 2.2) can now be built. Fig. 9 recapitulates the part of the φ, f, F, and H functions in such a construction. Let us give an example. Let μ be the membership function of a fuzzy subset "directional spatial relations between points." To assess the directional relationships between two raster objects A and B, Miyajima and Ralescu [16] normalize the histogram of angles associated with (A,B) and assimilate it to a fuzzy subset. This fuzzy subset is then matched to the one defined by μ, using the compatibility notion [3]. Let H^M_μ be the H function so defined. It is easy to show [14] that H^M_μ and μ share property [P12] and that H^M_μ satisfies properties [P13] and [P14]. According to Propositions 3, 4, and 8, for any real number r, the family of relations generated by F_r and H^M_μ therefore, shares property [A1] with μ and satisfies properties [A2] to [A4]. In Section 5, two specific families are considered: those generated by F_0 and H^M_μ and by F_2 and H^M_μ.

5 Experimental Results and Comparative Study

In this section, we compare three ways to assess the directional spatial relations between two areal objects. The first method, **M**, has been proposed by Miyajima and Ralescu in [16]. It relies on the construction of a histogram of angles. In the other methods, **M0** and **M2**, the previous construction is replaced by the construction of F-histograms. One histogram results from function f_0 and the other from function f_2. Remember that f_0 satisfies [P5] and f_2 [P6]. The aim is not to present a deep comparative study involving the different directional spatial relations that have been defined in the literature. Note that the assessment of the directional relations between two objects does not always require, even implicitly, a representation of the relative position of these objects (e.g., [1]). The aim is not to analyze, defend, or criticize the way the histograms are processed by **M**, **M0**, and **M2** either. It is to demonstrate, whatever the way chosen, that the notion of the histogram of forces generalizes and may supersede that of the histogram of angles.

Methods	**M**	**M0**	**M2**
Histograms	of angles $\mathcal{A}^{AB}$	of forces F_0^{AB}	of forces F_2^{AB}
Handling of histograms	by H^M_μ (see IV.B) with μ such that [16]: $\forall x \in [0,\pi/2]$, $\mu(x) = \cos^2(x)$		

Only raster data can be handled by **M**. In addition, processing nondisjoint objects is not explicitly tackled in the literature. Last, the manipulation of fuzzy objects is always reduced to that of their level-cuts, which are crisp objects (e.g., it is demonstrated in [14] that the calculation scheme used in [16] corresponds to the *simple sum scheme* described in Appendix A). For these reasons, the test images presented in Fig. 14 are numerical images and involve disjoint crisp objects.

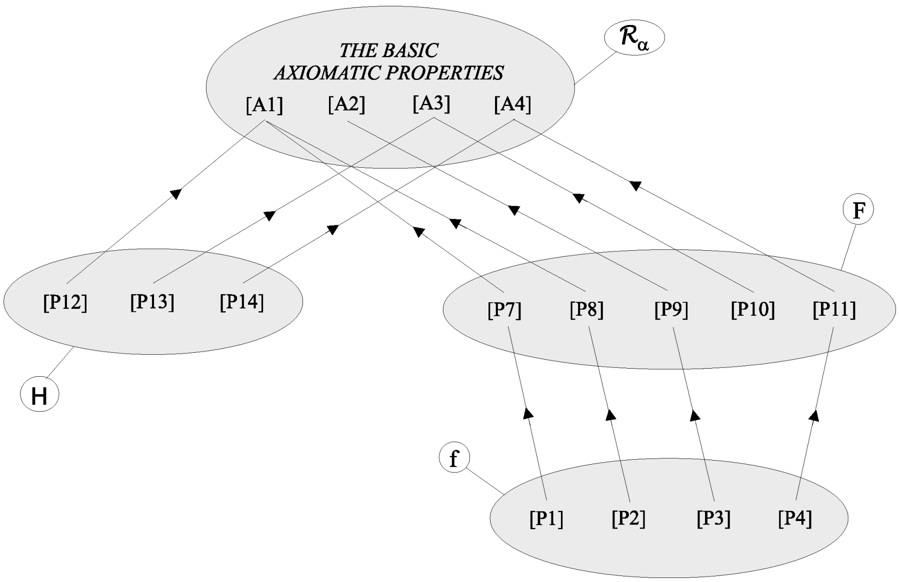


Fig. 15. Principal links between the properties [P1] to [P14].

5.1 Complexity and Computation Time

The data structure used to represent objects gives them certain characteristics. Taking these characteristics into account with the exploitation of the properties of F and the power of integral calculus allows fast and efficient F-histogram computation. The process of an assessable couple (A, B) of crisp objects is translated into a set of assessments of predetermined algebraic expressions. Each assessment corresponds to the process of a couple of trapeziums in the case of vector data (Fig. 10a), a couple of segments (more precisely, a batch of couples of pixels) in the case of raster data (Fig. 10b). In the first case (vector data), for any direction θ considered, the objects are partitioned by sorting both A and B vertices, following direction $\theta + \frac{\pi}{2}$. The complexity of methods **M0** and **M2** is $\mathcal{O}(n \cdot log(n))$, where n denotes the total number of object vertices. In the second case (raster data), the partitioning of the objects is based on the rasterization of a group of parallel lines by means of Bresenham's algorithm in integer arithmetic [19]. The maximum complexity of methods **M0** and **M2** is then $\mathcal{O}(n\sqrt{n})$, where n denotes the number of pixels of the processed image. Note that the complexity of **M** is $\mathcal{O}(n^2)$. As the number of segments in the longitudinal sections of the objects increases, the possibility to batch process the pixel couples decreases, as does the efficiency of **M0** and **M2**. But, even for a very noisy image with very distorted objects, numerous pixel couples can be batch processed and the **M0** and **M2** methods remain very efficient in comparison with **M**.

In practice, a histogram of forces is represented by a limited number of values; the set of directions is made discrete. The computation time of an F-histogram is obviously proportional to the number of directions which are considered. Experience shows that it is judicious to set the digitization step between two and three degrees (Fig. 11). With a lower step, the computation time gets higher but the results achieved by **M0** and **M2** only differ by 10^{-3}.

The different methods were implemented in C programming language on a 100MHz Sparc 4 without any excessive attention to optimization. We tested the methods on a series of eight homothetic images (i.e., related by a dilatation). As shown in Fig. 12, the CPU time necessary for the computation of the histogram of angles quickly becomes prohibitory. The larger the objects are, the greater the number of pixel couples to be considered and the less efficient **M**. For instance, image 4—170×160 pixels —was processed in about two CPU seconds by **M0** and **M2**, and 150 by **M**. The homothetic image of $2 \times 170 \times 160$ pixels was processed in about four seconds by **M0** and **M2**, and 600 by **M**!

5.2 Results

The results achieved by methods **M** and **M0** are comparable. This is not surprising. The histograms produced by the **M** and **M0** methods are fundamentally equivalent. But **M** prefers the horizontal and vertical directions, while **M0** does not (Fig. 13). **M2**, which directly takes metric information into account, produces specific histograms. For some simple configurations, the results achieved by **M** and **M2** are also comparable. For others, they are not at all. Which method provides the "best" results? The answer obviously depends on the application considered. We just deal here with what Gapp [6] has called the *basic meanings* of spatial relations (the model proposed by Gapp to define the semantics of spatial relations distinguishes *context-specific conceptual knowledge* from the *basic meanings* of the relations). However, note that, even when the results provided by **M2** are completely different from the others (see images 2, 3, 5, 7, and 8), **M2** expresses an opinion which is fully rational. Moreover, *no method* relying on the construction of a histogram of angles can express such an opinion. Indeed, histograms of angles do not take into account metric information.

6 Conclusion

In this paper, the notion of the histogram of forces has been introduced. It provides a fuzzy qualitative representation of the relative position between two areal objects. It generalizes and may supersede that of the histogram of angles habitually used. The notion of the histogram of forces offers solid theoretical guarantees. It allows explicit and modulable accounting of metric information. This is expressed by the choice of a numerical function with a real variable (φ). Different families of histograms can, therefore, be considered. One of them coincides with that of the histogram of angles, but without its weaknesses (long processing times, anisotropy, requirement for raster data, etc.). The histogram of forces is a powerful tool of representation with numerous potential applications. There are many opportunities for exploiting this tool, in particular, for defining directional spatial relations in better harmony with human perception, are clearly present in the literature [8], [9], [11], [16].

APPENDIX A

Properties [P1] to [P14]

The properties [P1] to [P6] are defined for any function f from $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ into $\mathbb{R}_+$ (Section 3.2.2).

[P1] f is defined and null on

$$\{(x, y, z) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ / x + y + z < 0\}$$

[P2] f is defined on $\mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}_+$ and has a strictly positive value on $\mathbb{R}_+^{*3}$. Moreover, $\forall (x_1, x_2, y, z) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}_+$,

$$f(x_1 + x_2, y, z) = f(x_1, y, z) + f(x_2, x_1 + y, z)$$

[P3] Let (x, y, z) be an element of $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$. If f is defined at (x, y, z) then it is defined at (z, y, z) and: $f(z, y, x) = f(x, y, z)$

[P4] There exists a map g from $\mathbb{R}_+^*$ into $\mathbb{R}$, such that, for any element (x, y, z) of $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ and for any element k of $\mathbb{R}_+^*$, if f is defined at (x, y, z), then it is defined at (kx, ky, kz) and: $f(kx, ky, kz) = g(k).f(x, y, z)$

[P5] Let (x, y_1, y_2, z) be an element of $\mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+$. If f is defined at (x, y_1, z), then it is defined at (x, y_2, z) and: $f(x, y_2, z) = f(x, y_1, z)$

[P6] Let (x, y, z) be an element of $\mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}_+$ and k an element of $\mathbb{R}_+^*$. If f is defined at (x, y, z), then it is defined at (kx, ky, kz) and: $f(kx, ky, kz) = f(x, y, z)$

The properties [P7] to [P11] are defined for any function F from T into $\mathbb{R}_+$ (Section 3.2.3).

[P7] Let θ be a real and let I and J be two disjoint segments such that (θ, I, J) belongs to T. Function F is defined at (θ, I, J). Moreover, if there exists no element (M, N) of $I \times J$ such that M is in direction θ of N, then $F(\theta, I, J)$ is null;

$$\{\forall (M, N) \in I \times J,\ \mathcal{R}_\theta(M, N) = 0\} \Rightarrow F(\theta, I, J) = 0$$

[P8] Let (θ, I', J') be an element of T and let I and J be two disjoint segments such that (θ, I, J) belongs to T. If I' and J' are included in I and J, respectively, then F is defined at (θ, I', J') and: $F(\theta, I', J') \leq F(\theta, I, J)$

[P9] Let (θ, I, J) be an element of T. If F is defined at (θ, I, J), then it is defined at $(\theta + \pi, J, I)$ and: $F(\theta + \pi, J, I) = F(\theta, I, J)$

[P10] Let (θ, I, J) be an element of T and sym a $\Delta_\beta(v)$-axis orthogonal symmetry. If F is defined at (θ, I, J), then it is also defined at $(2\beta - \theta, sym(I), sym(J))$ and:

$$F(2\beta - \theta, sym(I), sym(J)) = F(\theta, I, J)$$

[P11] There exists a map G from $\mathbb{R}_+^*$ into $\mathbb{R}_+$ such that, for any element (θ, I, J) of T and for any dilatation dil with a strictly positive ratio k, if F is defined at (θ, I, J), then it is defined at $(\theta, dil(I), dil(J))$ and:

$$F(\theta, dil(I), dil(J)) = G(k).F(\theta, I, J)$$

Property [P12] expresses a link between a map H from $Map([-\pi, \pi], \mathbb{R}_+)$ into $[0, 1]$ and a map μ from $\mathbb{R}$ into $[0, 1]$. The properties [P13] and [P14] are defined for any map H from $Map([-\pi, \pi], \mathbb{R}_+)$ into $[0, 1]$ (Section 4.2).

[P12] For any real δ and any strictly positive real η, there exists an element ϵ of $]0, \pi[$, such that, $\forall h \in Map([-\pi, \pi], \mathbb{R}_+)$,

$$(\bar{h} \neq 0 \text{ and } \bar{h}_{|[\delta-\pi, \delta-\epsilon] \cup [\delta+\epsilon, \delta+\pi]} = 0) \Rightarrow |H(h) - \mu(\delta)| < \eta,$$

where $\bar{h}$ represents the function defined on $\mathbb{R}$, with period 2π, whose restriction to $[-\pi, \pi]$ is h. In the formula, we deal with $\bar{h}$ when it is focused around δ.

[P13] $\forall (h_1, h_2) \in Map([-\pi, \pi], \mathbb{R}_+)^2$,

$$\{\forall \theta \in [-\pi, \pi], h_1(-\theta) = h_2(\theta)\} \Rightarrow H(h_1) = H(h_2)$$

[P14] $\forall (h_1, h_2) \in Map([-\pi, \pi], \mathbb{R}_+)^2$,

$$\{\exists K \in \mathbb{R}_+^* / h_1 = K.h_2\} \Rightarrow H(h_1) = H(h_2)$$

APPENDIX B

HANDLING of FUZZY OBJECTS

Definition 5. A fuzzy object *is a fuzzy subset E of the plane such that any α-cut E^α, with α an element of $[0, 1]$, is a crisp object. Let E be a fuzzy object and θ and v two reals: $E \cap \Delta_\theta(v)$ is a fuzzy subset of the plane denoted $E_\theta(v)$ and named a* longitudinal section *of E.*

The notion of the histogram of forces can easily be extended to fuzzy objects. Consider a function F defined for handling longitudinal sections of crisp objects. Let n be a nonnull positive integer and $(\alpha_i)_{i \in 1..n+1}$ a strictly decreasing sequence of reals such that $\alpha_1 = 1$ and $\alpha_{n+1} = 0$. Let (A, B) be a couple of fuzzy objects, whose membership functions take their values in $\{\alpha_i\}_{i \in 1..n+1}$. How can a histogram of forces be associated with (A, B)? Reduction of fuzzy subset processing to level-cut processing is a frequent practice. Several ways have been proposed in the literature. For example, we can use the schemes described below. The first derives directly from the generic scheme proposed by Dubois and Jaulent [2], the other from Krishnapuram et al. [11]. For any element i of $1..n$, value m_i denotes the difference $\alpha_i - \alpha_{i+1}$.

The double sum scheme

$$F(\theta, A_\theta(v), B_\theta(v)) = \sum_{i=1}^{n} \sum_{j=1}^{n} m_i m_j F(\theta, (A_\theta(v))^{\alpha_i}, (B_\theta(v))^{\alpha_j}) .$$

It is easy to show that: $F^{AB} = \sum_{i=1}^{n} \sum_{j=1}^{n} m_i m_j F^{A^{\alpha_i} B^{\alpha_j}}$.

The simple sum scheme

$$F(\theta, A_\theta(v), B_\theta(v)) = \sum_{i=1}^{n} m_i F(\theta, (A_\theta(v))^{\alpha_i}, (B_\theta(v))^{\alpha_i}).$$

It is easy to show that: $F^{AB} = \sum_{i=1}^{n} m_i F^{A^{\alpha_i} B^{\alpha_i}}$

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