Understanding the Spatial Organization of Image Regions by Means of Force Histograms: A Guided Tour

Pascal Matsakis

Computer Engineering and Computer Science Dept.
University of Missouri-Columbia
Columbia, MO 65211, USA
MatsakisP@missouri.edu

Abstract. Understanding the spatial organization of regions in images is a crucial task, essential to many domains of computer vision. The histogram of forces—a quantitative representation of the relative position between two objects—constitutes a powerful tool dedicated to this task. It encapsulates structural information about the objects as well as information about their spatial relationships. Moreover, it offers solid theoretical guarantees and nice geometric properties. Numerous applications have been studied, and new applications continue to be explored. For instance, force histograms can be compared through similarity measures for fuzzy scene matching. They can be used for describing relative positions in terms of spatial relationships modeled by fuzzy relations. They can also be used for scene description, where relative positions are represented by linguistic expressions. This chapter reviews and classifies work on the histogram of forces. It touches topics as varied as human-robot communication and spatial indexing mechanisms for medical image databases.

Keywords. Force histograms, relative positions, spatial relations, shape matching, scene matching, scene description, fuzzy sets, fuzzy logic, pattern recognition, image analysis, computer vision.

1 Introduction

Understanding the spatial organization of regions in images is a crucial task, essential to countless domains of computer vision. The notion of the histogram of forces, which was first introduced in [16] with the aim of providing new definitions of directional relations (such as “to the right of,” “above,” “to the west of,” “behind”), constitutes a powerful tool for accomplishing this task.

A force histogram is a quantitative representation of the relative position between two 2D objects. It encapsulates structural information about the objects as well as information about their spatial relationships. It is sensitive to the shape of the objects, their orientation and their size. It is also sensitive to the distance between them. In fact, the notion of the histogram of forces allows explicit and variable accounting of metric information. Moreover, it offers solid theoretical guarantees and nice geometric properties, ensures fast and efficient processing of vector data as well as of raster data, and enables the handling of fuzzy objects as well as of crisp objects, intersecting objects as well as of disjoint objects, and unbounded objects as well as of bounded objects.

The applications of the histogram of forces are numerous. So far, they seem to fall into three categories. The applications of the first category make "low-level" use of the histogram, i.e., the relative position between two objects is directly represented by the histogram associated with these objects. The typical application consists in comparing histograms through similarity measures for object pair matching. The question is whether a pair of objects (i.e., the two objects and their spatial relationships) corresponds (up to some geometric transformations, like translation, rotation and scaling) to another pair of objects. These two pairs may come from the same image, or from different images. Object pair matching leads to object matching (when the objects in a given pair are the same), shape matching, and scene matching (when many pairs of object pairs are considered). This is discussed in Section 3. The applications of the second category make "intermediate-level" use of the histogram of forces, i.e., the relative position between two objects is described in terms of a few spatial relationships. These relationships are represented by fuzzy spatial relations, and their evaluation relies on the computation of the histogram associated with the objects. The goal then is to assess specific spatial relationships (e.g., "to the right of"), or to compare the relative position of two objects with the relative position of two other objects (knowing that the objects themselves might be all different). This is discussed in Section 4. Finally, the applications of the third category make "high-level" use of the force histogram. The relative position between two objects is represented by words, i.e., linguistic expressions. These expressions are generated from the histogram associated with the objects, typically through the fuzzy spatial relations mentioned above. This is discussed in Section 5. First of all, in the following section, we present the notion of the histogram of forces and its fundamental geometric properties.

2 The Notion of the Histogram of Forces

In this paper, unless otherwise specified, the term "object" denotes a bounded, 2D crisp object (a rigorous definition of this term for the use of histograms of forces is given in [16,21]). We will return to this matter at the end of Section 2.1.

2.1 Description

The relative position of an object *A* with regard to another object *B* is represented by a function F^{AB} from $\mathbb{R}$ into $\mathbb{R}_+$. For any direction θ, the value $F^{AB}(\theta)$ is the total weight of the arguments that can be found in order to support the proposition "*A* is in direction θ of *B*." More precisely, it is the scalar resultant of elementary forces. These forces are exerted by the points of *A* on those of *B*, and each tends to move *B* in direction θ (Fig. 1). If F^{AB} is defined on $\mathbb{R}$, i.e., if for any θ the scalar resultant $F^{AB}(\theta)$ is finite, then the pair *(A,B)* is termed *F-assessable* and F^{AB} is called the *histogram of forces associated with (A,B) via F*, or the *F-histogram associated with (A,B)*. The object *A* is the *argument*, and the object *B* the *referent*.

Actually, the letter *F* denotes a numerical function. Let *r* be a real. If the elementary forces are in inverse ratio to d^r, where *d* represents the distance between the points considered, then *F* is denoted by F_r. The F_0-histogram (histogram of constant forces) and F_2-histogram (histogram of gravitational forces) have very different and very interesting characteristics. The former provides a global view of the situation. It considers the closest parts and the farthest parts of the objects equally, whereas the F_2-histogram focuses on the closest parts.

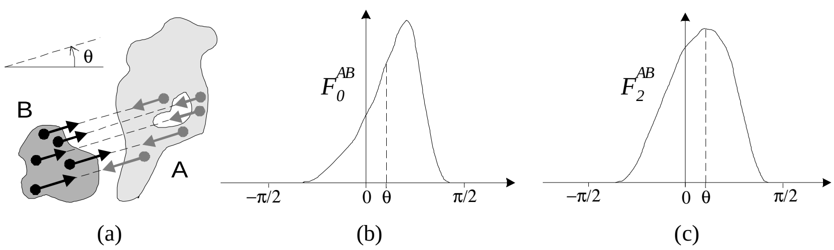

Fig. 1. Force histograms. (a) $F^{AB}(\theta)$ is the scalar resultant of forces (black arrows). Each one tends to move *B* in direction θ. (b) The histogram of constant forces associated with *(A,B)*. It represents the position of *A* relative to *B*. (c) The histogram of gravitational forces associated with *(A,B)*. It is another representation of the relative position between *A* and *B*.

It is shown [16,21] that for any *r*, any pair of disjoint objects is F_r-assessable. If *r* is lower than 1, any pair of overlapping objects is F_r-assessable too. The constraint on *r* can be bypassed by defining histograms of hybrid forces [16,24,33], but then, some geometric properties are lost. As mentioned at the very beginning of Section 2, the term "object" denotes here a bounded, 2D crisp object. Note however that F_r-histograms can also handle unbounded objects (if *r* is greater than 1 [16,24]), and fuzzy objects (this is discussed in [16,21]). In theory, they can handle neither 0D objects, nor 1D objects. In practice, this is usually not a limitation, since points and lines can easily be assimilated to 2D objects (see, e.g., Fig.

8(e)). Finally, vector data can be processed as well as of raster data [16,21,33]. In the first case (vector data), the complexity of force histogram computation is $O(nN\log(N))$, where N denotes the total number of object vertices and n the number of directions in which forces are computed (usually between 32 and 360, depending on the application that is considered). In the second case (raster data), the complexity is $O(nN\sqrt{N})$, where N denotes the number of pixels of the processed image. This complexity drops to $O(nN)$ for convex objects. Force histogram computation benefits from the power of integral calculus, is highly parallelizable, and utilizes a well-known algorithm that is commonly circuit coded in visualization systems.

2.2 Properties

Force histograms have nice geometric properties. Consider two objects A and B. Assume that (A,B) is F_r-assessable. The following properties hold. Properties 1 to 3 are illustrated by Fig. 2.

Property 1: The pair (B,A) is also F_r-assessable and:
$\forall\theta\in\mathbb{R},\ F_r^{BA}(\theta) = F_r^{AB}(\theta-\pi)$.

Property 2: Let sym be a Δ-axis orthogonal symmetry, and let α be the angle between the X-axis and Δ. The pair $(sym(A),sym(B))$ is F_r-assessable and:
$\forall\theta\in\mathbb{R},\ F_r^{sym(A)sym(B)}(\theta) = F_r^{AB}(2\alpha-\theta)$.

Property 3: Let dil be a central dilation[1] with a positive ratio λ.
The pair $(dil(A),dil(B))$ is F_r-assessable and:
$\forall\theta\in\mathbb{R},\ F_r^{dil(A)\,dil(B)}(\theta) = \lambda^{3-r}\,F_r^{AB}(\theta)$.

Property 4: Let $stre$ be an X-axis orthogonal stretch with a positive ratio k. For any real θ, let $\bar{\theta}$ be the value $atan(k^{-1}\tan\theta)$ if $\cos\theta$ is positive, the value θ if $\cos\theta$ is zero, and the value $atan(k^{-1}\tan\theta)+\pi$ otherwise. The pair $(stre(A),stre(B))$ is F_r-assessable and: $\forall\theta\in\mathbb{R},\ F_r^{stre(A)stre(B)}(\theta) = k^{2-r}\,[1+(k^2-1)\cos^2\theta]^{(r-1)/2}\ F_r^{AB}(\bar{\theta})$.

Properties 2 and 3 define the behavior of the F_r-histograms towards any similarity transformation. For instance, Property 2 implies Properties 5 and 6 below. The stretch in Property 4 is particular, since its axis is the X-axis, and its ratio is positive. However, the properties 2 and 4 define the behavior of the F_r-histograms towards any orthogonal one-way stretch. All proofs are in [16] (Chapter 2, Appendix A), and [18]. Note that stretches are not similarity transformations.

Property 5: Let $tran$ be a translation. $(tran(A),tran(B))$ is F_r-assessable and:
$\forall\theta\in\mathbb{R},\ F_r^{tran(A)tran(B)}(\theta) = F_r^{AB}(\theta)$.

Property 6: Let rot be a ρ-angle rotation. $(rot(A),rot(B))$ is F_r-assessable and:
$\forall\theta\in\mathbb{R},\ F_r^{rot(A)rot(B)}(\theta) = F_r^{AB}(\theta-\rho)$.

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[1] A *dilation* is also known as a *homothecy* (or *homothety*).

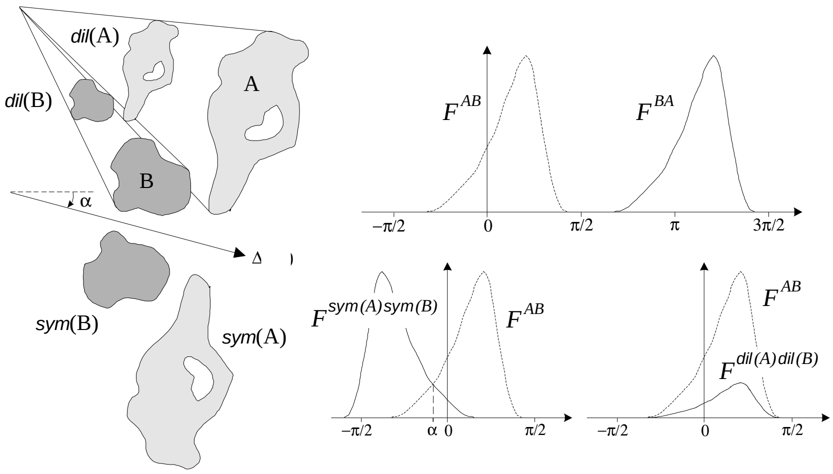

Fig. 2. Some properties of force histograms.
Knowing F^{AB}, it is easy to retrieve F^{BA}, $F^{sym(A)sym(B)}$ and $F^{dil(A)dil(B)}$.

2.3 Inverse Problem

The histogram of forces is sensitive to the shape, the orientation of and the distance between the objects it is associated with. One may then wonder if different pairs of objects can lead to the same histogram. Consider, for instance, two disjoint objects *A* and *B*. Let $\mathcal{O}_r^{AB}$ be the set of object pairs *(A',B')* such that $F_r^{A'B'}=F_r^{AB}$. It is clear that *(A,B)* belongs to $\mathcal{O}_r^{AB}$. Moreover, it is not the only element of $\mathcal{O}_r^{AB}$. Consider any translation *tran*, any π-angle rotation *rot*, and any dilation *dil*. According to Properties 1, 3, 5 and 6, the pairs *(tran(A),tran(B))* and *(tran(rot(B)),tran(rot(A)))* also belong to $\mathcal{O}_r^{AB}$; if *r* is equal to 3, the pairs *(tran(dil(A)),tran(dil(B)))* and *(tran(rot(dil(B)),tran(rot(dil(A)))* belong to $\mathcal{O}_r^{AB}$ too. Does $\mathcal{O}_r^{AB}$ contain other elements than these ones? It is an intricate problem that remains to be solved. However, in practice, if two object pairs *(A,B)* and *(A',B')* are such that $F_r^{A'B'}$ is equal to F_r^{AB}, then *(A',B')* is most probably one of the pairs listed above. A more detailed discussion on this topic can be found in [18].

3 Comparing Force Histograms

Some applications of the histogram of forces make "low-level" use of the histogram. In these applications, histograms are compared through similarity measures.

3.1 Principle

Consider four objects A_1, B_1, A_2 and B_2 in the Euclidean plane (e.g., A_1 and B_1 come from the segmentation of some digital image I_1, and A_2 and B_2 from the segmentation of another image I_2). Assume there exists a central dilation dil with a positive ratio λ such that $A_2=\mathit{dil}(A_1)$ and $B_2=\mathit{dil}(B_1)$ (e.g., I_1 and I_2 represent the same physical objects, but have different scaling factors). Let m_1 be the mean of $F^{A_1B_1}$, let m_2 be the mean of $F^{A_2B_2}$, and let μ be some similarity measure (e.g., the classic sigma-count of the intersection over the union [18]). According to Property 3, the dilation ratio λ (i.e., the scaling factor ratio) is $[m_2/m_1]^{1/(3-r)}$. Moreover, the value $\mu(F^{A_2B_2}, (m_2/m_1)F^{A_1B_1})$ tells us about the validity of the assumption concerning the existence of dil. Similarly, Property 6 allows us to check the existence of a rotation rot such that $A_2=\mathit{rot}(A_1)$ and $B_2=\mathit{rot}(B_1)$, and to retrieve the rotation angle (or *azimuth difference*). Property 4 allows us to handle the case where the projection plane of the camera (or *image plane*) is not parallel to the observed plane. The declination of the camera platform (or *tilt*) can even be retrieved. Naturally, it is possible to consider combinations of the geometric transformations involved in the different properties. The histogram of forces therefore constitutes a powerful tool for object pair matching. This is thoroughly discussed in [18]. Note that when $A_1=B_1$ and $A_2=B_2$ (the histograms $F^{A_1B_1}$ and $F^{A_2B_2}$ are then called *F-signatures*), object pair matching corresponds to object matching. Hence, the notion of the histogram of forces can also be exploited in pattern recognition and classification problems. This has been illustrated in [23,36,22] (Fig. 3). Finally, the force histogram can be of great use in scene matching, which is one obvious application of object pair matching. In Section 3.2, we examine scene matching in LADAR (Laser Radar) imagery, and exploit the fact that force histograms are able to handle fuzzy objects.

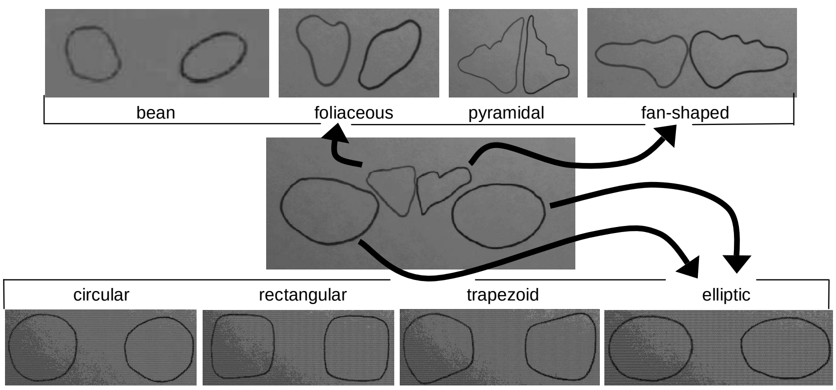

Fig. 3. Experts distinguish four models of sinuses (top) and four models of orbits (bottom). The orbits and sinuses represented by drawings from craniums of the 3rd century A.D. (center) can be classified using the histogram of forces [23,36,22].

3.2 Application to Fuzzy Scene Matching

Consider, for instance, a range image generated by the laser radar system mounted on a surveillance plane. We show in [15] that it is possible to manipulate the three-dimensional data contained within the image and to create a version of the scene as seen from above. In the transformed view, each object is represented by a fuzzy region, and the spatial relationship between two objects is represented by a force histogram. Matching two scenes then comes down to comparing force histograms. Each comparison gives a degree of similarity between the two object pairs under consideration, as well as an assessment of the pose parameters. These values can finally be combined to find the correct scene matching.

First, the range image (Fig. 4(a)) is “lighted.” A normal to each pixel is calculated from the three-dimensional positions in the range data of the pixel and its immediate neighbors. It allows an intensity value to be associated with the pixel. The processed scene looks more natural to the human eye (Fig. 4(b)). In [15], a hand segmentation is then performed (Fig. 4(c)). It results in a rough approximation of the objects’ edges. The inaccuracies in the segmentation are corrected using range information: each approximated edge is automatically adjusted to correlate with the best possible “real” edge on the range data.

The tilt of the camera platform is assessed using a Hough-like transform. Two consecutive data points in any of the columns of the range data allow a candidate tilt angle to be produced. The most commonly found angle is assumed to be the actual tilt (thus, the method only works on data which represents a relatively flat landscape with objects). A similar method can be used to find the roll of the scene.

The labels of the segmented image are then mapped to the three-dimensional positions of the range data, and these positions are rotated by the tilt angle. The resulting image is the scene as viewed from above (Fig. 4(d)). It has many gaps in it. Some are due to the general spreading of the pixels caused by the rotation. They are easy to handle. The other gaps correspond to uncertainty areas, i.e., areas which were obscured by objects in the original image. They are “filled” by associating each object with a fuzzy region (Fig. 4(d)). To best represent the uncertainty, different types of boundaries are considered, depending on whether the gap occurs on the front side of the object (i.e., the side closest to the camera), or on the back side of it.

Once the two scenes to match have been transformed to a declination independent angle, a force histogram is calculated for each individual fuzzy region pair. At this point, when comparing two histograms not coming from the same scene, the rotational (i.e., azimuth) difference and scaling ratio of the images can be assessed and varied to maximize a given similarity measure. Rotational difference is calculated as the distance between the centroids of the two histograms, and is varied by shifting one of the histograms along the horizontal axis. The scaling ratio is calculated by comparing the histogram averages, and is varied by stretching one of the histograms vertically. This is justified by properties 6 and 3.

In [15], experiments were conducted on a pair of LADAR range images provided by the Naval Air Warfare Center. The two images represent a power-plant complex seen from two different viewpoints. As illustrated by Fig. 5, we focused on a set of four buildings. Our goal was to coherently label the buildings, and to retrieve the declination angles and the rotational difference. Twelve histograms were computed (6 per scene), and 6!=720 possible scene matches were considered. For each possible matching, an overall “matching degree” was derived from the computed maximum similarity measures, scaling ratios, and rotational differences. The

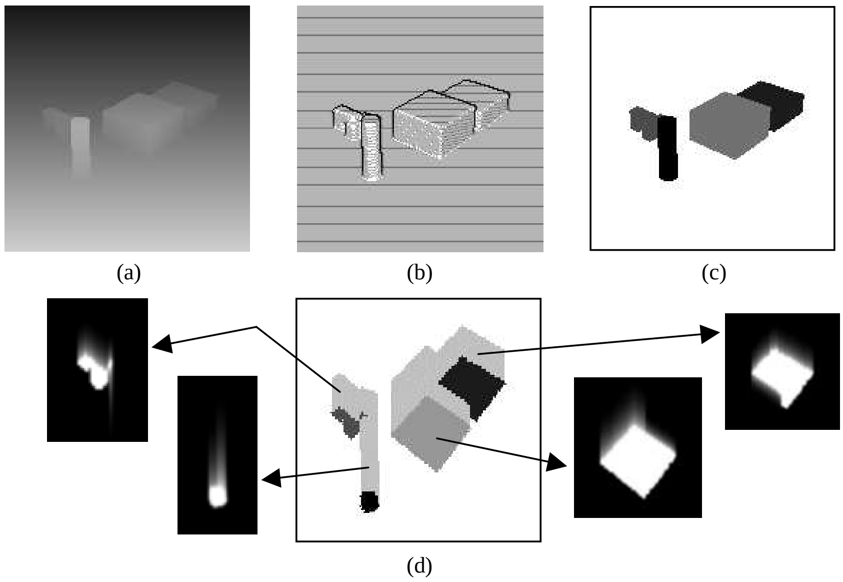

Fig. 4. By manipulating the three-dimensional data contained within a range image, it is possible to create a version of the scene as seen from above [15]. In the transformed view, each object is represented by a fuzzy region. (a) Range image. (b) Lighted scene. (c) Segmented image. (d) Overhead view and fuzzy regions.

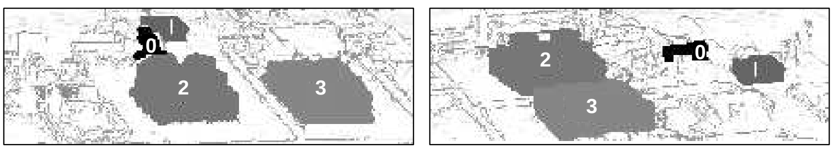

Fig. 5. These two images represent a power-plant complex seen from two different viewpoints. A fuzzy scene matching approach based on force histogram computation makes it possible to coherently label the buildings, and to retrieve the declination angles and the rotational difference.

matching with the highest degree was found to be the true matching and led to a coherent labeling of the buildings. The actual pose parameters were not provided by the NAWC and could not be compared with the recovered values. However, extensive experiments on synthetic data have shown that the tilts can be recovered to within 5º and the azimuth difference to within 10º [15,18].

4 Defining Fuzzy Spatial Relations

In [5], Freeman proposed that the relative position of two objects be described in terms of spatial relationships. He also proposed that fuzzy relations be used, because "all-or-nothing" standard mathematical relations are clearly not suited to models of spatial relationships. Freeman's ideas were widely adopted. But many authors assimilated 2D objects to very elementary entities such as a point (centroid) or a (bounding) rectangle. This approach is extremely practical, therefore it has often been used, notably for spatial reasoning and representation and processing of qualitative spatial knowledge. However, it cannot be hoped to give a satisfactory modeling of the relationships, because a lot of morphological information on the considered objects is lost. We show here that the histogram of forces—which encapsulates a large amount of information on the objects[2]—lends itself, with great flexibility, to the definition of fuzzy spatial relations. This is the "intermediate-level" use of the force histogram.

4.1 Directional Relations

Relative position is often assimilated to directional relationships. The point of view is improper, but it shows the importance of these relationships in the field of computer vision. A *family of fuzzy directional spatial relations* is a series $(\mathcal{R}_\alpha)_{\alpha\in\mathbb{R}}$ of fuzzy binary relations. The relation $\mathcal{R}_\alpha$ reads "in direction α of." Depending on α and on context, it may also read "to the right of," "above," "to the west of," "in front-left of," etc. It connects any pair of spatial entities A and B with a numerical value[3]. This value, $A\mathcal{R}_\alpha B$, corresponds to the degree of truth of the proposition "A is in direction α of B." It is a real number greater than or equal to 0 (proposition completely false) and less than or equal to 1 (proposition completely true). Entity A is the *argument* of the proposition and entity B the *referent*. Points are easy to handle (Fig. 6), but the problem gets complex when parameters such as shape, size and orientation are involved [29,25]. Although numerous methods for defining families of directional relations between 2D objects can be found in the literature, few of these methods simultaneously meet the following requirements:

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[2] The histogram of forces does not constitute, of course, the only way to preserve structural information (see, for instance, [2] and [25]).

[3] The pair may also be connected with a confidence interval or a fuzzy number.

(a) No object is assimilated to an elementary entity such as a point or a rectangle.
(b) The defined relations are fuzzy relations, and not "all-or-nothing" ones.
(c) The defined family satisfies the basic axiomatic properties [16,21] which are—in a more or less explicit way—widely adopted by computer scientists: *(i)* two objects can be assimilated to points if they are distant enough; *(ii)* the directional relations are not sensitive to scale; *(iii)* neither a space dimension nor a direction are preferred; *(iv)* the semantic inverse principle [5] is respected (e.g., *A* is to the left of *B* as much as *B* is to the right of *A*).

The centroid method (see, e.g., [10]) and the methods described in [7,9] do not meet requirement **(a)**; the ones described in [1,6,13] do not meet requirement **(c)**. Actually, as far as we are aware, the only methods that fairly meet the previous requirements are based—explicitly or not—on the notion of the histogram of angles presented in [25]. These methods are the compatibility method [25], the aggregation method [14], the possibility method proposed in [2][4] (but not the necessity method, neither the average one), and, to a certain extent, the neural network methods [11]. In [16,19], we showed that the corresponding families of directional relations can be advantageously redefined using force histograms instead of angle histograms. In [16,20], we noted that most families of fuzzy relations run counter to the fact that, generally, people do not combine more than two spatial prepositions when translating visual information into natural language descriptions [8,27]. We also exhibited a coherent and rational perception of the world that no existing family could model.

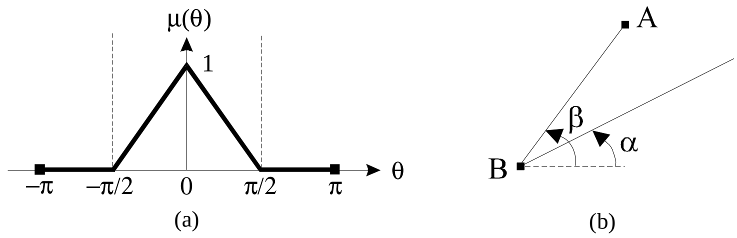


Fig. 6. Example of directional relations between points. (a) A typical fuzzy set. (b) The degree of truth of the proposition "*A* is in direction α of *B*" is $\mu(\beta-\alpha)$.

These facts led us to introduce alternative families based on the notion of the histogram of forces [16,20]. The idea is to impose physical considerations on the histograms. Let r be a real and (A,B) an F_r-assessable pair of objects. Our goal is to assess the degree of truth of a proposition like "*A* is in direction α of *B*," where α represents any angle. Here, we will only consider the proposition "*A* is in direction 0 of *B*," which will be read "*A* is to the right of *B*." For another value of α,

[4] Although its definition is based on a morphological and fuzzy pattern matching approach, the possibility degree introduced in [2] is basically a function of the histogram of angles.

you can simply perform the computations described below on the shifted histogram, $F_r^{AB}(\theta+\alpha)$. The forces exerted on B are classified in different types. First, the set of directions is divided into four quadrants as shown in Fig. 7. The forces $F_r^{AB}(\theta)$ of the outer quadrants ($\theta\in[-\pi,-\pi/2]\cup[\pi/2,\pi]$) are elements which, to various degrees, weaken the proposition "*A* is to the right of *B*"; the forces of the inner quadrants ($\theta\in[-\pi/2,0]\cup[0,\pi/2]$) are elements which support the proposition. Some forces of the third quadrant are used to compensate—as much as possible—the contradictory forces of the fourth one. The proportion of these compensatory forces is defined by some angle θ_+. Forces of the second quadrant are used in a similar way to compensate the contradictory forces of the first one. The amount of these compensatory forces is defined by θ_-. The remaining forces are called the effective forces. A threshold τ divide them into optimal and sub-optimal components. The optimal components support the idea that *A* is "perfectly" to the right of *B*: whatever their direction, they are regarded as horizontal and pointing to the right. The "average" direction $\alpha_r(RIGHT)$ of the effective forces is then computed, in conformity with this agreement.

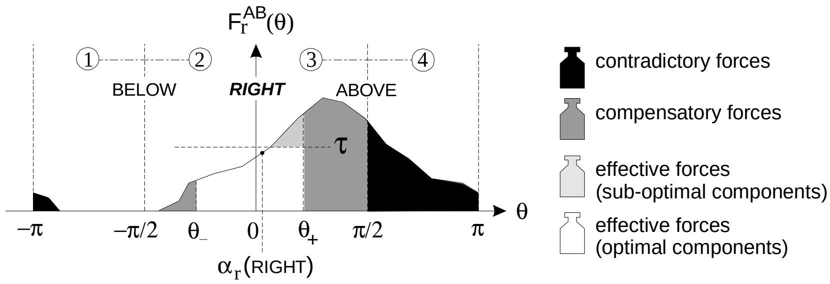
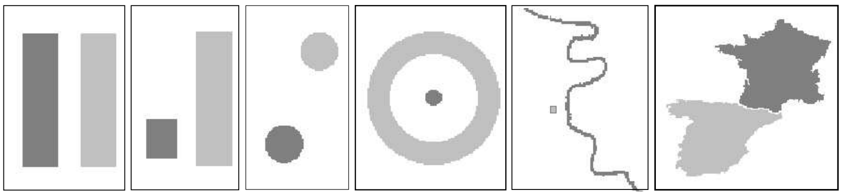


Fig. 7. Force typology associated with the proposition "*A* is to the right of *B*."

Finally, the degree of truth of "*A* is to the right of *B*" is set to $\mu(\alpha_r(RIGHT))\times\overline{a}_r(RIGHT)$. In this expression, $\overline{a}_r(RIGHT)$ denotes the percentage of the effective forces (i.e., the sum of the effective forces divided by the sum of all forces), and μ is the membership function of a fuzzy set that can be employed to define a family of fuzzy directional relations between points. In our experiments, we used the typical triangular function graphed in Fig. 6(a). The most optimistic point of view consists in saying that any effective force is optimal, i.e., $\tau=+\infty$. Then, $\mu(\alpha_r(RIGHT))\times\overline{a}_r(RIGHT)$ is equal to $\overline{a}_r(RIGHT)$—since $\alpha_r(RIGHT)$ is 0 and $\mu(0)$ is 1. The most pessimistic point of view consists in saying that any effective force is sub-optimal, i.e., $\tau=0$. In that case, the expression $\mu(\alpha_r(RIGHT))\times\overline{a}_r(RIGHT)$ gives some value $\underline{a}_r(RIGHT)$. Setting τ to the average—or a weighted average—of the effective forces constitutes a natural compromise. The degree of truth of "*A* is to the right of *B*" is then found to be some value $a_r(RIGHT)$. The 3-tuple $(\underline{a}_r(RIGHT), a_r(RIGHT), \overline{a}_r(RIGHT))$ defines a triangular fuzzy number. It corresponds to the histogram's "opinion" regarding the proposition "A is to the right of B." According to F_r^{AB}, the degree of truth of that proposition is $a_r(RIGHT)$, the maximum degree of truth that can reasonably be attached to it (say, by another source of information) is $\overline{a}_r\,(RIGHT)$, and the minimum degree that can reasonably be attached to it is $\underline{a}_r(RIGHT)$. The method has been presented in detail in [20]. The French-speaking reader is also invited to consult [19] (or [16]). Fig. 8 shows six pairs of objects. For each pair (A,B), four propositions have been assessed, using two families of fuzzy directional spatial relations. The four propositions are "A is to the right of B," "A is above B," "A is to the left of B" and "A is below B." The two families, **F0** and **F2**, are based on the construction of F_0 and F_2-histograms, and the distinction between contradictory, compensatory and effective forces, as described above. The degrees of truth produced by **F0** and **F2** are displayed in Table 1. Different comparative studies can be found in [16,21,19].



(a) (b) (c) (d) (e) (f)

Fig. 8. Six pairs of objects. The referent is drawn darker than the argument.

TABLE 1

F0 and **F2**'s opinions regarding the relative position of the objects displayed in Fig. 8 (the degrees of truth are in hundredths). According to the two families, an object cannot be simultaneously a bit to the left and a bit to the right of another. In particular, as illustrated by case (d), directional relationships should not substitute for surroundedness (see Section 4.2.1). Also note that **F2** sometimes conflicts with **F0** (i.e., $a_2 > \overline{a}_0$ or $a_2 < \underline{a}_0$), and vice versa (e.g., case of the house and the river). This can be exploited at a higher level (see Section 5).

F0	$\underline{a}_0$	a_0	$\overline{a}_0$	$\underline{a}_0$	a_0	$\overline{a}_0$	$\underline{a}_0$	a_0	$\overline{a}_0$	$\underline{a}_0$	a_0	$\overline{a}_0$	$\underline{a}_0$	a_0	$\overline{a}_0$	$\underline{a}_0$	a_0	$\overline{a}_0$
RIGHT	100	**100**	100	69	**88**	100	23	**23**	100	0	**0**	0	0	**0**	0	0	**0**	0
ABOVE	0	**0**	0	31	**38**	58	77	**87**	100	0	**0**	0	0	**0**	0	0	**0**	0
LEFT	0	**0**	0	0	**0**	0	0	**0**	0	0	**0**	0	82	**87**	92	44	**54**	83
BELOW	0	**0**	0	0	**0**	0	0	**0**	0	0	**0**	0	18	**19**	21	56	**76**	99

(a) (b) (c) (d) (e) (f)

F2	$\underline{a}_2$	a_2	$\overline{a}_2$	$\underline{a}_2$	a_2	$\overline{a}_2$	$\underline{a}_2$	a_2	$\overline{a}_2$	$\underline{a}_2$	a_2	$\overline{a}_2$	$\underline{a}_2$	a_2	$\overline{a}_2$	$\underline{a}_2$	a_2	$\overline{a}_2$
RIGHT	100	**100**	100	82	**100**	100	23	**23**	100	0	**0**	0	0	**0**	0	0	**0**	0
ABOVE	0	**0**	0	18	**21**	37	77	**87**	100	0	**0**	0	0	**0**	0	0	**0**	0
LEFT	0	**0**	0	0	**0**	0	0	**0**	0	0	**0**	0	95	**98**	99	24	**29**	39
BELOW	0	**0**	0	0	**0**	0	0	**0**	0	0	**0**	0	5	**5**	7	76	**93**	98

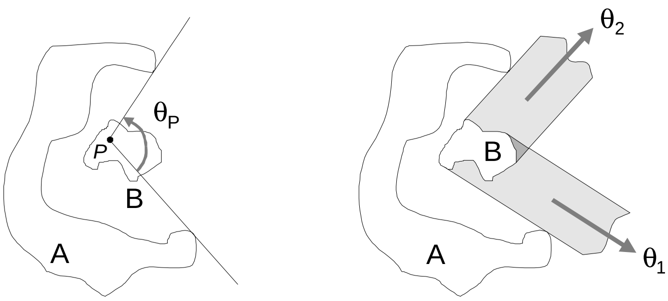
4.2 Other Spatial Relations

The histogram of forces was introduced with the aim of providing new definitions of directional relations [16,21], and its use for the modeling of other relationships has been the subject of little investigation. It is clear that the force histogram does not constitute an ideal representation of the relative position between objects, i.e., it does not enable the definition of "any" fuzzy spatial relation. Nevertheless, it can be employed to model a variety of relationships—although it might mean working under certain assumptions on the objects, or using extra geometric features. This is what we illustrate here with two examples.

4.2.1 Surroundedness

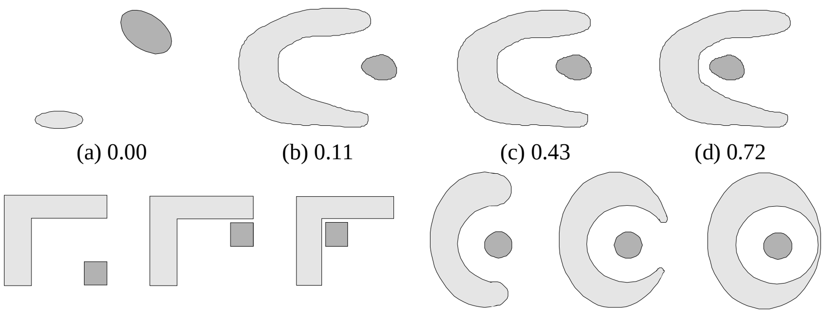
Surroundedness can be considered a particular case of *separation* [28]. It is an important spatial relationship in the interpretation of a scene, and many quantitative definitions have been proposed. There are two main approaches. The first approach relies on the fact that according to most families of fuzzy directional relations an object can be in many directions with respect to another. As mentioned in Section 4.1, this feature is questionable: usually, people do not combine more than two spatial prepositions when translating visual information into natural language descriptions [8,27]. However, some authors [25,2] support the idea that it allows "surrounds" (and "is surrounded by") to be derived. Knowing that *A* is somewhat above, below, to the right and to the left of *B* as well, one could conclude that *A* surrounds *B*. In fact, drawing such a conclusion is not reasonable, unless it is known that the argument *A* does not intersect the convex hull of *B*. In other words, "*A* surrounds *B*" can be assessed only if it is known that *B does not surround A at all*. The reason is that the directional relations are tied by the *semantic inverse* principle [5] (e.g., *A* is to the left of *B* as much as *B* is to the right of *A*). Therefore, without constraints on the objects, there is no way to know which one surrounds (or includes!) the other. The second approach derives from Rosenfeld's visual surroundedness [30]. It is based on the computation of a histogram of angles. It supposes that the argument *A* is connected and does not intersect *B*. For any pixel *P* of *B*, let θ_P be the angle made by the two tangents from *P* to *A* as in Fig. 9(a). To each element θ of $[0,2\pi]$, the histogram associates the number of pixels *P* such that θ_P is equal to θ. In [35], the degree of truth for "*A* surrounds *B*" is produced by a multilayer perceptron fed by the histogram values and trained on aggregate responses from a panel of people. Other authors resort to a decreasing membership function μ from $[0,2\pi]$ into [0,1]. The function μ is chosen such that $\mu(\theta)$ is 1 if θ is 0, and is 0 if θ is greater than π. In [26], the histogram of angles is assimilated to a fuzzy set and matched to μ, using the compatibility notion [4]. The degree of truth for "*A* surrounds *B*" is obtained as the center of gravity of the compatibility fuzzy set. In [14], the histogram is used to compute the aggregated value (e.g., the arithmetic mean, or the generalized mean [12]) of the $\mu(\theta_P)$, when *P* describes *B*. The degree of truth for "*A* surrounds *B*" is set to this value. Compared with the first one, the second approach gives definitions of surroundedness that are much more consistent with human perception [35]. However, the methods proposed are computationally expensive, and vector data cannot be handled.

We show in [33] that the histogram of forces can easily be employed to assess surroundedness. Let μ be a membership function as above, and let $[\theta_1,\theta_2]$ be the largest interval—with θ_1 in $]-\pi,\pi]$—on which the force histogram F^{AB} is zero. Fig. 9(b) illustrates the meaning of angles θ_1 and θ_2. The degree of truth for "*A* surrounds *B*" is set to $\mu(\theta_2-\theta_1)$. Once again, the argument is supposed to be connected. Moreover, it should not intersect the convex hull of the referent (like in the first approach). The definition is quite simple, but compares with the others [33]. Some examples are shown in Fig. 10.



(a) (b)

Fig. 9. Defining surroundedness.
(a) Use of a histogram of angles. (b) Use of a histogram of forces.



(a) 0.00 (b) 0.11 (c) 0.43 (d) 0.72

(e) 0.00 (f) 0.23 (g) 0.47 (h) 0.32 (i) 0.84 (j) 1.00

Fig. 10. Surroundedness based on the histogram of forces. In our experiments, μ was linear: for any θ in $[0,\pi]$, $\mu(\theta)=1-\theta/\pi$. The value 1.00 means that the proposition "*A* surrounds *B*" is assessed to be completely true, and the value 0.00 that it is completely false (where *A* denotes the light gray object and *B* the dark one). Similar images are used in [35] for training and testing the network-based method, and in [33] for comparing different methods.

The advantages of the force histogram-based method are that it ensures faster processing of raster data, and it is able to handle vector data as well. Also, the directional relations can be assessed concurrently. Note that the degree of truth for "*A* surrounds *B*" does not really depend on the histogram values. The only thing that matters is which values are equal to zero and which ones are not. This has two important consequences. First, the results do not depend on the choice of the force histogram. Second, the method is not extremely robust. Slight changes in the object shapes (especially at the two "ends" of the argument) may have a noticeable impact on the degrees of truth. In fact, similar comments can be addressed to any method that derives from Rosenfeld's visual surroundedness: the results are not sensitive to the thickness of the argument, only to tangency points. The force histogram-based definition is exploited in Section 5.2.2. The application described there does not suffer from the above-mentioned limitations (constraints on the objects, low robustness). However, other applications might. In [17], the degree of truth for "*A* surrounds *B*" is redefined using the force histogram values (and not only the fact that these values are either zero or non-zero). Related relationships, like "between" and "among," are also examined. Designing a new type of histogram of forces constitutes another promising avenue. The idea would be to adopt a novel set of axiomatic properties, and to change the way the longitudinal sections are handled [16,21].

4.2.2 Inner-Adjacency

Adjacency is another important spatial relationship between image regions. Different quantitative definitions have been proposed, notably in [37,30,3]. Our work on spatial indexing mechanisms for medical image databases has led us to consider a particular relationship called "inner-adjacency." In [31], the position of an object *A* relative to another object *B* is represented by the histogram of constant forces associated with (*A*,*B*–*A*). Some histogram values may thus be zero even when *A* and *B* intersect. The degree of truth of the proposition "*A* is inner-adjacent to *B*" is set to

$$max\left(\frac{i}{b}, min\left(\frac{i}{a}, 1-\frac{hmin}{hmean}\right)\right),$$

where *a* denotes the size (area) of the argument, *b* the size of the referent, *i* the size of $A \cap B$, and *hmin* and *hmean* are the histogram minimum and average values. Fig. 11 shows 8 schematic configurations and the corresponding degrees of truth.

The fuzzy spatial relation presented here models one clinically meaningful relationship. It was implemented in the spatial indexing mechanism of the medical content-based image retrieval system proposed in [31]. The system was tested using a set of 2,080 HRCT lung images (Fig. 12). It achieved a 90% accuracy rate for lesion retrieval based on inner-adjacency. Spatial relationships between lesions and anatomical landmarks in medical images are critically important in disease diagnosis.

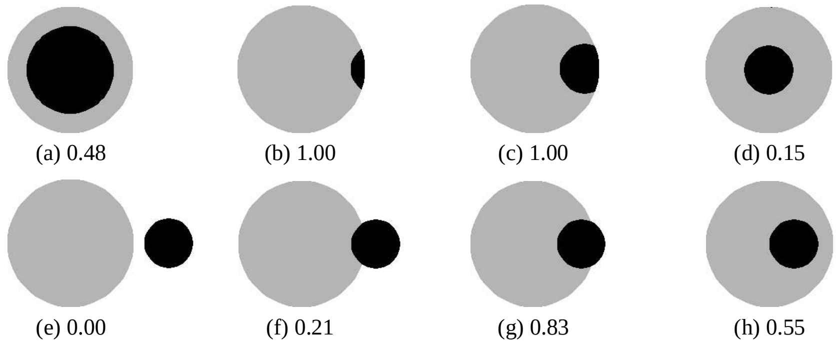


Fig. 11. Inner-adjacency. Each value is the degree of truth of the proposition "*A* is inner-adjacent to *B*," where *A* denotes the black object and *B* the intersected gray one.

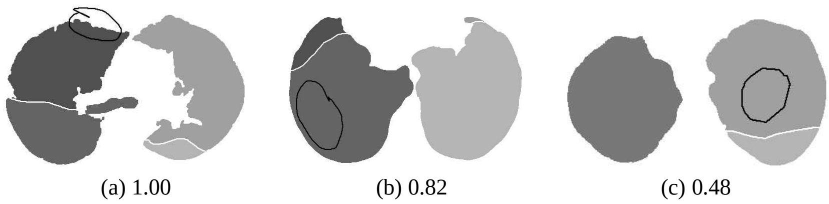


Fig. 12. Lesion retrieval based on inner-adjacency. The three images were generated from HRCT (High Resolution Computed Tomography) lung images. Consider (a). The two lungs are clearly visible. Fissures divide them into chambers. A lesion has been identified by the physician (note that the region enclosed by the line outside the lung is not part of the lesion). In (a)(b)(c), we are interested in the position of the lesion relative to the chamber it belongs to. The inner-adjacency decreases from left to right.

5 Generating Linguistic Spatial Descriptions

High-level computer vision applications hold a great potential for fuzzy set theory because of its links to natural language. Linguistic scene description, a language-based interpretation of regions and their relationships, is one such application that is starting to bear the fruits of fuzzy set theoretic involvement. In this section, we show how the fuzzy relations presented in Section 4 can be utilized to produce logical linguistic spatial descriptions along with assessments as to the validity of the descriptions. This is the "high-level" use of the histogram of forces.

5.1 Principle

In [20], a linguistic description of the relative position between any 2D objects *A* and *B* is generated from F_0^{AB} (the histogram of constant forces associated with *(A,B)*) and F_2^{AB} (gravitational forces). As already mentioned in Section 2.1, these two histograms have very different and very interesting characteristics. The former, F_0^{AB}, provides a global view of the situation. It considers the closest parts and the farthest parts of the objects equally, whereas F_2^{AB} focuses on the closest parts.

The linguistic description output by the system in [20] relies on the sole primitive directional relationships: "to the right of," "above," "to the left of" and "below" (imagine that the objects are drawn on a vertical surface). First, the histograms' opinions regarding these relationships are computed (see Section 4.1). For instance, the following triangular fuzzy numbers are extracted from F_2^{AB}: ($\underline{a}_2(RIGHT)$, $a_2(RIGHT)$, $\overline{a}_2\,(RIGHT)$), ($\underline{a}_2(ABOVE)$, $a_2(ABOVE)$, $\overline{a}_2\,(ABOVE)$), ($\underline{a}_2(LEFT)$, $a_2(LEFT)$, $\overline{a}_2\,(LEFT)$) and ($\underline{a}_2(BELOW)$, $a_2(BELOW)$, $\overline{a}_2\,(BELOW)$). For each one of the four primitive directions (say, *RIGHT*), F_0^{AB} may consider that F_2^{AB}'s opinion is defensible ($\underline{a}_0(RIGHT) \leq a_2(RIGHT) \leq \overline{a}_0\,(RIGHT)$), or is not defensible ($a_2(RIGHT) < \underline{a}_0(RIGHT)$ or $a_2(RIGHT) > \overline{a}_0\,(RIGHT)$), and vice versa. The histograms' opinions are combined, as illustrated in Fig. 13. Six features result from this combination: two primitive directions (a *primary* direction and a *secondary* direction), and four numeric values (a degree of truth and a measure of agreement are associated with each direction). They feed a fuzzy rule base that produces the expected linguistic description. The system handles a set of 16 adverbs (like "mostly," "perfectly," etc.) which are stored in a dictionary, with other terms, and can be tailored to individual users. A description is generally composed of three parts. The first part involves the primary direction (e.g., "*A* is mostly to the right of *B*"). The second part supplements the description and involves the secondary direction (e.g., "but somewhat above"). The third part indicates to what extent the four primitive di-

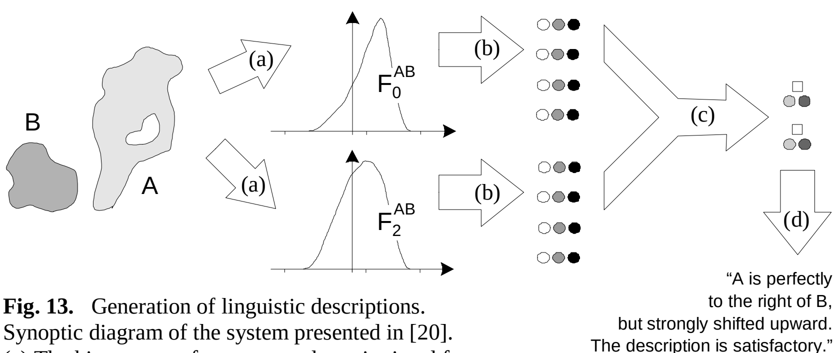


Fig. 13. Generation of linguistic descriptions. Synoptic diagram of the system presented in [20]. (a) The histograms of constant and gravitational forces are computed. (b) Each histogram gives its opinion about the relative position between the objects. (c) The two opinions are combined. (d) A fuzzy rule base outputs the description.

rectional relationships are suited to describing the relative position of the objects (e.g., "the description is satisfactory"). In other words, it indicates to what extent it is necessary to turn or not to other spatial relations. For instance, if the self-assessment is "not satisfactory," then the system proposed in [33] turns to "surrounds," using the force histogram-based method presented in Section 4.2.1. All details can be found in [20] and [33].

5.2 Application to Image Scene Description

The system for linguistic scene description developed in [20] was tested on numerous synthetic and real data examples. In particular, we used it to describe the relative position between regions from LADAR (Laser Radar) range images of the power-plant at China Lake, CA. These images were provided by the Naval Air Warfare Center. They were processed by applying first a median filter, and then a pseudo-intensity filter. Finally, the filtered images were segmented and labeled manually. Here, contrary to what is said in Section 3.2, range information was not utilized to correct the inaccuracies in the segmentation. Fig. 14 shows some pairs of objects (or groups of objects) that were examined in our experiments.

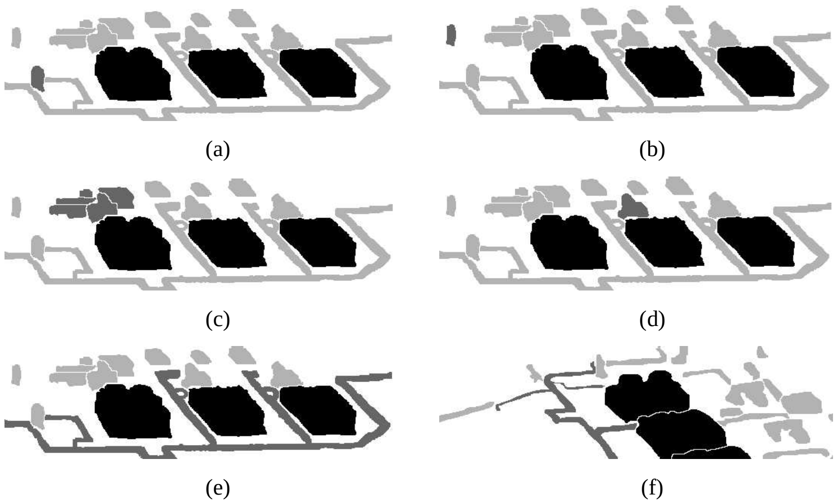
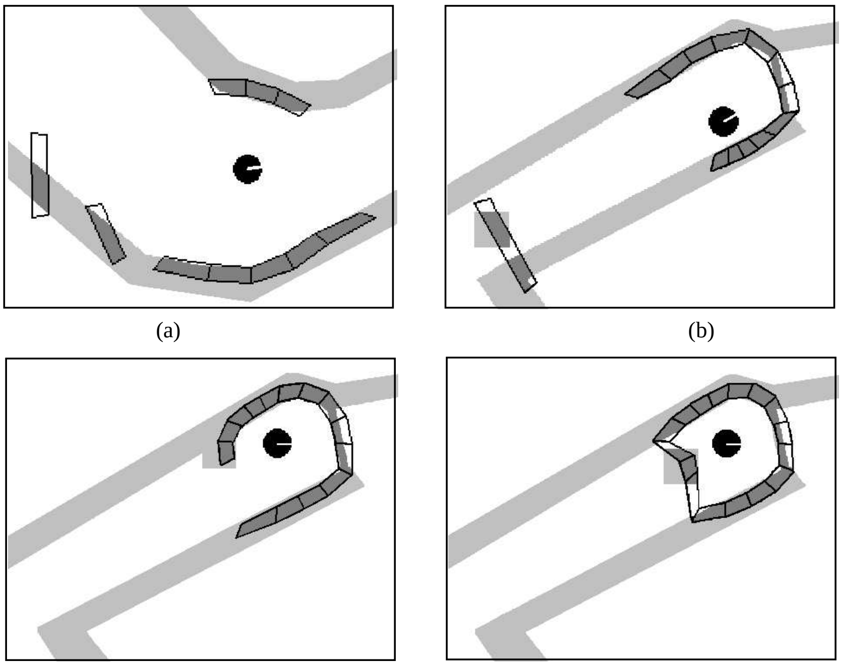


Fig. 14. (a) "The tower (in dark gray) is perfectly to the left of the stackbuildings (in black). The description is satisfactory." (b) "The tower is to the left of the stackbuildings, but a little above. The description is satisfactory." (c) "The group of storehouses is loosely above-left of the stackbuildings. The description is satisfactory." (d) "The storehouse is perfectly above the stackbuildings, but slightly shifted to the left. The description is satisfactory." (e) "???????" (f) "The pipe is loosely to the left of the stackbuildings, but slightly shifted downward. The description is rather satisfactory."

The richness of the system's language is generally very well employed. Consider Fig. 14(c): the system notes that the relationship is not a perfect above-left, and uses the adverb "loosely" to indicate a bias in one direction. Now, consider Fig. 14(d): the system points out that the storehouse is slightly shifted to the left. In some cases, no pertinent description relying on the sole primitive directional relationships can be given, and the message "???????" is generated. For instance, the relative position between the pipe and the stackbuildings of Fig. 14(e) cannot be described. The output is appropriate, since surroundedness is not considered in [20]. Although the system globally performs very well and produces good intuitive results, some descriptions are not totally satisfactory. Dealing with a rich language is tricky, i.e., it is always easier to be right when vague and imprecise. Consider Fig. 14(f). A piece of the pipe extends between the uppermost and middle stackbuildings. At the end of the extension, the pipe has a strong "downward" relationship with the uppermost building (and a weak "upward" relationship with the middle one). As a result, the argument is assessed to be slightly shifted downward relative to the referent. Note, however, that a good amount of ambiguity is detected, and the system itself considers the description *rather* satisfactory.

5.3 Application to Human-Robot Communication

In [33], we show how linguistic expressions can be generated to describe the spatial relations between a mobile robot and its environment, using readings from a ring of sonar sensors (see also [32] and [34]). Our work is motivated by the study of human-robot communication for non-expert users. The eventual goal is to utilize these linguistic expressions for navigation of the mobile robot in an unknown environment, where the expressions represent the qualitative state of the robot with respect to its environment, in terms that are easily understood by humans. The differences between the systems described in [20] and [33] are few, but they are not inconsiderable. In [20], the force histograms are computed from raster data, and the spatial reference frame is implicitly determined by the reader's location (*world* view). In [33], the histograms are computed from vector data, and the reference frame is determined by the intrinsic orientation of the robot (*egocentric* view). The two works therefore complement one another. They illustrate the fact that the histogram of forces is able to handle vector data as well as of raster data, and makes it easy to switch between a world view and an egocentric view. The system in [33] also considers surroundedness, using the fuzzy relation presented in Section 4.2.1. Since the reference object is always the robot (a Nomad 200 with 16 sonar sensors evenly distributed along its circumference), the limitations mentioned in that section are not an issue (the robot is convex and is not supposed to jam itself in the walls). Moreover, the system expresses proximity information (based on the sonar readings). Two levels of abstraction are provided, and detailed individual descriptions can be combined into more synthetic descriptions. On the other hand, the language used to describe directional positions is coarser than the one in [20] (and the dictionaries are slightly different). The reason is that only a rough representation of the environment objects can be built anyway. Consider Fig. 15(a). The robot is heading rightwards in an angled hallway. There are ten sonar returns (i.e., ten sensors return non-maximum range values). Each one gives a trapezoid, located in the corresponding sonar cone, and built using a constant arbitrary depth. There is a question on whether adjacent sonar readings are from a single object or multiple objects. If the robot cannot fit between the object parts that are responsible for two adjacent sonar readings, then we consider these parts to be from the same obstacle (and the trapezoids are linked). Even if there are actually two objects (this happens in Fig. 15(d), right behind the robot), they may be considered as one for robot navigation purposes. If the robot can fit, we consider separate obstacles (the trapezoids are not linked). The two readings may come from the same object, but there is no way to know that until the robot gets closer and we have a better resolution of the object (since more sensors would



(a) (b) (c) (d)

Fig. 15. A robot describes its environment. (a) "There is an obstacle on my right; it extends forward; it is very close. There is one on my left; it extends forward; it is very close. One is behind me; it is close. Another one is mostly behind me, but somewhat to the right; it is close." (b) "There is an obstacle that surrounds me on the front. There is another one behind me; it is far." (c) "There is an obstacle that surrounds me, but there is an opening on the rear-right." (d) "I am surrounded."

detect its presence). In Fig. 15(a) for instance, two obstacles are detected behind the robot, although the readings come from the same wall. Note that the caption only shows the detailed individual descriptions. In the higher level of abstraction, the robot describes its environment as follows: “There is an obstacle on my right, one on my left, two behind me.” The three other figures, Fig. 15(b) to Fig. 15(d), illustrate how the system handles surroundedness, using the fuzzy relation presented in Section 4.2.1.

The experiments were carried out with the Nomad simulator. The program runs at real-time speed. Processing of all obstacles (i.e., construction of the polygonal representations, computation of the force histograms and generation of the linguistic descriptions) is done faster than the robot can move, so there are no delayed results.

6 Conclusion

We have shown in this chapter that the notion of the histogram of forces can be of great use in understanding the spatial organization of image objects. This is a crucial problem, essential to countless domains of computer vision. The histogram of forces provides a fuzzy qualitative representation of the relative position between 2D objects. Because it offers solid theoretical guarantees and has nice geometric properties, it can be used in scene matching, and enables the pose parameters to be retrieved. It can also be exploited in pattern recognition. For instance, the F-signature—a particular force histogram that represents the shape of an object—has been used to classify cranium sinuses. Spatial databases can clearly benefit from a tool like the histogram of forces. We are currently working on new spatial indexing mechanisms for medical image databases. They rely on the computation of force histograms for modeling the relationships between lesions and anatomical landmarks. Geographic information systems constitute a promising ground also, especially as force histograms are able to handle vector data in a very efficient manner. Moreover, the histogram of forces lends itself, with great flexibility, to the definition of numerous fuzzy spatial relations. In particular, new families of fuzzy directional relations have been introduced. They preserve important relative position properties, and can provide inputs to systems for linguistic scene description. One such system has been developed and dedicated to human-robot communication. Using readings from a ring of sonar sensors, a mobile robot describes its spatial relationship with the environment. The program runs at real-time speed. In the future, we plan to address two important challenges. The first one is very interesting from a purely theoretical point of view (probably less from a practical point of view, although it might be useful in scene matching for instance). It consists in solving the inverse problem, i.e., finding all the pairs of objects associated with a given force histogram. The second challenge consists in extending the notion of the histogram of forces so that three-dimensional entities can be handled. Analysis of 3D magnetic resonance images and design of virtual environments are two examples of potential applications. We also plan to define new fuzzy spatial relations, especially new models of "surrounds," and to develop mechanisms to adapt fuzzy relations and spatial descriptions to individual users.

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