Permutations with repetition

So far we’ve looked at permutations and combinations of objects where each object was distinct (distinguishable). Now we will consider allowing multiple copies of the same (identical/indistinguishable) item.

How many strings of length 10 can be formed from the english alphabet?

**solution:** $26^{10}$

Note that this is similar to a permutation where we allow the letters to be repeated.

**Theorem**
The number of $r$ – *permutations* on a set of $n$ objects with repetition allowed is $n^r$. 
Combinations with repetition

Example: The beer store.

You just finished your 2910 midterm - you think you did very well. On your way to the beer store, you find a $100 bill. Once at the beer store you find your 3 favorite beers are on sale for $24 a case: Bohemian, Red Stripe, and Pabst Blue Ribbon. It must be your lucky day. You decide to spend the $100 to buy 4 cases. You can’t help but wondering: How many ways can I choose 4 cases of beer from these brands?

Solution: Note, you are choosing cases with repetition allowed.

<table>
<thead>
<tr>
<th></th>
<th>4 Boh</th>
<th>4 RedS</th>
<th>4 PBR</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 Boh, 1 RedS</td>
<td>3 Boh, 1 PBR</td>
<td>3 RedS, 1 Boh</td>
<td></td>
</tr>
<tr>
<td>3 RedS, 1 PBR</td>
<td>2 Boh, 2 PBR</td>
<td>2 RedS, 2 PBR</td>
<td></td>
</tr>
<tr>
<td>2 Boh, 2 RedS</td>
<td>2 PBR, 1 Boh, 1 RedS</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 Boh, 1 RedS, 1 PBR</td>
<td>2 PBR, 1 Boh, 1 RedS</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Thus, there are different ways to purchase 4 cases from these 3 brands.

Imagine if you had to determine the different ways to choose 4 cases from 20 different brands? An exhaustive list would not be feasible.

Example: Grabbing Cash.

How many different ways can you select 5 bills from a cash box containing $1, $2, $5, $10, $20, $50, and $100 bills?

Solution: \( C(11, 5) = \binom{11}{5} = 462 \)

Where did that number come from?
Combinations with repetition

Let’s go back to the beer store example. Let B represent Bohemian, S represent red Stripe and P represent Pabst.

How many ways can one choose \( r = 4 \) cases of beer from \( n = 3 \) brands?

\[
\begin{align*}
BBBB & \rightarrow \text{****} | | \\
BBBS & \rightarrow \text{***} | * | \\
BBBP & \rightarrow \text{***} | * | * \\
BBSS & \rightarrow \text{**} | ** | \\
BBSP & \rightarrow \text{**} | * | * |\\
BBPP & \rightarrow \text{**} | * | ** \\
BSSS & \rightarrow \text{*} | ** | * \\
\ldots & \ldots
\end{align*}
\]

Observe that the \( n - 1 = 2 \) bars are used to partition off the different brands of beer, and the \( r = 4 \) *’s are used to represent the chosen cases.

**Note:** the number of such strings correspond to the number of length 6 binary strings with 4 (or 2) ones.

Theorem

There are \( C(n + r - 1, r) = \binom{n+r-1}{r} \) ways to choose \( r \) items from a set of \( n \) elements when repetition is allowed.

Example: Grabbing Cash.

How many different ways can you select 5 bills from a cash box containing $1, $2, $5, $10, $20, $50, and $100 bills?

**Solution:** There are \( n - 1 \) “bars” required to partition the \( n \) different bills. Placing \( r \) *’s to represent the bills, we end up with strings of length \( n + r - 1 \) containing \( r \) *’s.

In this case \( n = 7 \) and \( r = 5 \), so there are \( C(11,5) = 462 \) ways to grab 5 bills.
Examples

How many ways can we place 7 identical balls into 8 separate (but distinguishable) boxes?

Solution: \( n = 8 \) and \( r = 7 \), so there are \( \binom{8+7-1}{7} = \binom{14}{7} \) ways to do this.

How many different strings (order matters!) can be made by reordering the letter of the word MISSISSIPPI?

Not quite a permutation with repetition since there are limited number of each letter.

How many ways can you place the 4 S’s in the string of length 11? \( \binom{11}{4} \).

Now in the 7 remaining spots, how can we place the 2 P’s?: \( \binom{7}{2} \).

Now in the 5 remaining spots, how can we place the 4 I’s?: \( \binom{5}{4} \).

Now in the 5 remaining spots, how can we place the 1 M?: \( \binom{5}{1} \).

\[
\frac{11!}{7!4!} \cdot \frac{7!}{5!2!} \cdot \frac{5!}{4!1!} \cdot \frac{11!}{4!2!4!1!} = 11!
\]

Permutations with indistinguishable objects

Theorem A

Consider \( k \) symbols where there are \( n_i \) occurrences of the symbol \( i \) where \( 1 \leq i \leq k \). Let \( n \) equal the sum \( \sum_{i=1}^{n} n_i \). Then the number of different permutations is:

\[
\frac{n!}{n_1! \cdot n_2! \cdots n_k!}
\]

How many different strings can be made using 4 A’s, 3 B’s, 7 C’s and 1 D?

Apply the previous theorem. There are \( k = 4 \) symbols where \( n_1 = 4 \), \( n_2 = 3 \), \( n_3 = 7 \) and \( n_4 = 1 \). Thus \( n = 4 + 3 + 7 + 1 = 15 \). The total number of such strings are:

\[
\frac{15!}{7! \cdot 4! \cdot 3! \cdot 1!}
\]
Distributing distinct objects into boxes

**Theorem B**

The number of ways to distribute $n$ distinct objects into $k$ distinguishable boxes so that there are $n_i$ objects placed into box $i$ is:

$$\frac{n!}{n_1! \cdot n_2! \cdots n_k!}$$

How many ways can we distribute a poker hand of 5 cards to four different players using a deck of 52 cards?

We must distribute 52 unique (distinguishable) cards to the 4 players, leaving the rest in another pile of 32. Thus, we have 5 boxes - 4 have size 5 and one has size 32 (the rest of the deck).

Applying the above theorem we get:

$$\frac{52!}{5! \cdot 5! \cdot 5! \cdot 5! \cdot 32!}$$

different ways to deal 4 hands of 5 cards.

Correlating the theorems

What is the correspondence between the last two theorems?

How many different strings can be made using 4 A’s, 3 B’s, 7 C’s and 1 D?

Consider the string of length 15 composed of A’s, B’s, C’s, and D’s from the earlier example. Instead of placing the letters into the 15 unique positions, we place the 15 positions into the 4 boxes which represent the 4 letters. Apply Theorem B.

How many ways can we distribute a poker hand of 5 cards to four different players using a deck of 52 cards?

Let each player and the deck be represented by 5 symbols: $P_1, P_2, P_3, P_4, D$. The number of occurrences of $P_1$ is 5 (he/she gets 5 cards). Same for $P_2, P_3, P_4$. The deck $D$ has 32 occurrences. Apply Theorem A to get the number of distinct strings composed of these symbols.

For example, consider a string of length 52: $DDDP_1DDDP_2P_3DDPP_1DDPP_1 \cdots$. Player 1 gets the card associated with positions 4, 11, 14 … which are predetermined.
Example

You are hosting a party and have invited 5 men and 9 women. How many ways can you seat your guests such that no two men sit together if

▶ you seat them in a line?
▶ you seat them at a round table?