Permutations and Combinations

CIS 2910

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Permutations

Definition

A permutation of a set of distinct objects is an ordered arrangement of these objects.

Example: All permutations of 1,2,3:

123, 132, 213, 231, 312, 321.

Given $n$ objects there are $n!$ different permutations. Remember: $0! = 1$

Definition

An ordered arrangement of $r$ elements of a $n$-set is called an $r$-permutation.

Example: All 3-permutations of 1,2,3,4:

123, 124, 132, 134, 142, 143, 213, . . . , 432.

The number of $r$-permutations of an $n$-set:

$$P(n, r) = n \cdot n - 1 \cdot n - 2 \cdot (n - r + 1) = \frac{n!}{(n - r)!}$$
Permutations

Example: In a horse race with 7 horses, how many different ways are there to select a trifecta (ordering of top 3 horses)?

\[ P(7, 3) = \frac{7!}{4!} = 7 \cdot 6 \cdot 5 = 210 \]

Example: How many permutations of ABCDEFGH contain the block ABC?

Since ABC must be together, we really have 6 elements to order: ABC,D,E,F,G,H. Thus there are 6! such permutations.

Combinations

Definition

An \( r \)-combination of an \( n \)-set is an unordered selection of \( r \) elements from the set. In other words, an \( r \)-combination is simply a subset of size \( r \).

The number of \( r \)-combinations of an \( n \)-set where \( 0 \leq r \leq n \):

\[ C(n, r) = \frac{n!}{r!(n-r)!} = \binom{n}{r} \quad \text{(spoken: \( n \) choose \( r \))} \]

How do we come up with this formula?

Proof

Observe that we can obtain all \( r \)-permutations by taking every \( r \)-combination and then ordering the elements of each combination in every possible way (\( r! \)). Thus:

\[ P(n, r) = C(n, r) \cdot r! \]

Now solve for \( C(n, r) \), plugging in the formula for \( P(n, r) \).
Combinations

Example: How many ways can we select two items from the set \{a,b,c,d\}?

$$C(4,2) = \binom{4}{2} = \frac{4!}{2! \cdot 2!} = \frac{24}{4} = 6$$

Observe, that when selecting these elements, the order doesn’t matter.

Thus, there are 6 different subsets of size 2, from a set of size 4.

Example: The worst hockey team is allowed to draft any three players from the top 3 teams. The top 3 teams have 20, 24, 22 players respectively. How many different ways can the worst team choose 3 players?

Observe, that the order of the players does not matter. So we are choosing 3 players from a pool of 66:

$$C(66,3) = \binom{66}{3} = \frac{66!}{63! \cdot 3!} = \frac{66 \cdot 65 \cdot 64}{3!} = 45,760$$

Combinations

How many ways are there to select (choose) \(x\) players from a 10-member tennis team to compete in an upcoming tournament, where \(0 \leq x \leq 10\)?

\[
\begin{align*}
  x = 0 : & \quad \binom{10}{0} = \frac{10!}{0!10!} = 1 \\
  x = 1 : & \quad \binom{10}{1} = \frac{10!}{9!1!} = 10 \\
  x = 2 : & \quad \binom{10}{2} = \frac{10!}{8!2!} = 45 \\
  x = 3 : & \quad \binom{10}{3} = \frac{10!}{7!3!} = 120 \\
  x = 4 : & \quad \binom{10}{4} = \frac{10!}{6!4!} = 210 \\
  x = 5 : & \quad \binom{10}{5} = \frac{10!}{5!5!} = 252 \\
  x = 6 : & \quad \binom{10}{6} = \frac{10!}{4!6!} = 210 \\
  x = 7 : & \quad \binom{10}{7} = \frac{10!}{3!7!} = 120 \\
  x = 8 : & \quad \binom{10}{8} = \frac{10!}{2!8!} = 45 \\
  x = 9 : & \quad \binom{10}{9} = \frac{10!}{1!9!} = 10 \\
  x = 10 : & \quad \binom{10}{10} = \frac{10!}{0!10!} = 1 \\
\end{align*}
\]

Do you see a pattern?
Combinations

**Theorem:** \( C(n, r) = C(n, n-r) \)

\[
C(n, n-r) = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!} = C(n, r)
\]

**Alternate proof:**

Let \( S \) be a set with \( n \) elements. Every subset \( A \) with \( r \) elements corresponds uniquely to a subset of size \( n-r \) elements (all those elements not in \( A \)).

Therefore the number of subsets of size \( r \) is the same as the number of subsets of size \( n-r \): \( C(n, r) = C(n, n-r) \).

The alternate proof above is an example of a **combinatorial proof**: it uses counting arguments, rather than algebraic manipulation.

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**Example 1**

The English alphabet has 21 consonants and 5 vowels. How many strings of 6 letters contain:

(a) exactly one vowel?
(b) exactly two vowels?
(c) at least one vowel?
(d) at least two vowels?

**Solution:** First, we know there are \( 26^6 \) strings with length 6 and \( 21^6 \) strings with only consonants.

(a) There are 6 positions to place 1 vowel, and 5 ways to select the vowel. This leaves 5 free positions for consonants: \( 6 \cdot 5 \cdot 21^5 \).

(b) There are \( \binom{6}{2} \) ways to select the 2 positions for the vowels, and 5 ways to pick each vowel. This leaves 4 positions free for the consonants: \( 15 \cdot 5^2 \cdot 21^4 \).

(c) Apply subtraction: the number of all strings minus the number with no vowel: \( 26^6 - 21^6 \).

(d) Apply subtraction: \( (c) - (a) = 26^6 - 21^6 - 6 \cdot 5 \cdot 21^5 \).
Example 2

DNA sequences are often represented as strings over the alphabet AGCT. How many DNA sequences of length 10 have the following properties?

(a) There are exactly 3 G’s?
(b) There are exactly 3 G’s and 3 T’s?
(c) There is a subsequence of the form TTT?
(d) There are exactly 3 G’s or 3 A’s?
(e) The first three letters are distinct.

Solution:

(a) The 3 G’s can be placed in \( \binom{10}{3} \) ways. The remaining 7 positions can be any of the other 3 letter. Thus: \( \binom{10}{3} \cdot 3^7 \).

(b) Place the G’s first: \( \binom{10}{3} \). Then place the T’s: \( \binom{7}{3} \). Fill the remaining 4 spots with A and C: \( 2^4 \). Thus: \( \binom{10}{3} \binom{7}{3} 2^4 \).

(c) There must be at least 3 T’s. Use subtraction: total strings minus strings with 0 T’s, 1 T, and 2 T’s: \( 4^{10} - 3^{10} - \binom{10}{1} 3^9 - \binom{10}{2} 3^8 \).

(d) Must apply inclusion/exclusion. If we count all strings with 3 G’s then add all strings with 3 A’s, we will count strings with both 3 G’s and 3 A’s twice.

(e) \( P(4,3) \cdot 4^7 \)

Example 3

Given an ordering of 7 playing cards \( c_1, c_2, \ldots c_7 \), how many such orderings are there such that:

(a) The first 4 cards are all kings?
(b) The first 4 cards contain 4 of a kind?
(c) There exists 4 kings?
(d) There exists 4 of a kind?
(e) Answer (c) and (d) if the order of the cards does not matter?
(f) Why does (d) get more difficult if we deal a hand of 8 or more cards?

Solution:

(a) There are 4! ways to order the kings: \( 4! \cdot P(48,3) = 4! \cdot 48 \cdot 47 \cdot 46 \)

(b) There are 13 choices for the 4 of a kind: \( 13 \cdot (a) \).

(c) Select spots for kings, then order the kings: \( \binom{7}{4} \cdot (a) \).

(d) \( 13 \cdot (c) \)

(e) \( (c): \binom{48}{3} \) \( (d) 13 \cdot \binom{48}{3} \)

(f) Must beware careful of double counting hands that have two four of a kind’s. Count hands with exactly one four of a kind, then add hands with 2 four of a kinds.