Permutations and Combinations

CIS 2910

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Permutations

Definition

A **permutation** of a set of distinct objects is an *ordered* arrangement of these objects.

Example: All permutations of 1,2,3:

$$123, 132, 213, 231, 312, 321.$$

Given n objects there are $n!$ different permutations. Remember: $0! = 1$

Definition

An ordered arrangement of r elements of a n-set is called an **r-permutation**.

Example: All 3-permutations of 1,2,3,4:

$$123, 124, 132, 134, 142, 143, 213, \ldots, 432.$$

The number of r-permutations of an n-set:

$$P(n,r) = n \cdot n-1 \cdot n-2 \cdot (n-r+1) = \frac{n!}{(n-r)!}$$

Permutations

Example: In a horse race with 7 horses, how many different ways are there to select a trifecta (ordering of top 3 horses)?

$$P(7,3) = 7!/4! = 7 \cdot 6 \cdot 5 = 210$$

Example: How many permutations of ABCDEFGH contain the block ABC?

Since ABC must be together, we really have 6 elements to order: ABC,D,E,F,G,H. Thus there are 6! such permutations.

Combinations

Definition

An **r-combination** of an n-set is an *unordered* selection of r elements from the set. In other words, an r-combination is simply a subset of size r.

The number of r-combinations of an n-set where $0 \leq r \leq n$:

$$C(n,r) = \frac{n!}{r!(n-r)!} = \binom{n}{r} \qquad \text{(spoken: } n \text{ choose } r\text{)}$$

How do we come up with this formula?

Proof

Observe that we can obtain all r-permutations by taking every r-combination and then ordering the elements of each combination in every possible way ($r!$). Thus:

$$P(n,r) = C(n,r) \cdot r!$$

Now solve for $C(n,r)$, plugging in the formula for $P(n,r)$.

Combinations

Example: How many ways can we select two items from the set {a,b,c,d}?

$$C(4,2) = \binom{4}{2} = \frac{4!}{2! \cdot 2!} = 24/4 = 6$$

Observe, that when selecting these elements, the *order doesn't matter*.

Thus, there are 6 different subsets of size 2, from a set of size 4.

Example: The worst hockey team is allowed to draft any three players from the top 3 teams. The top 3 teams have 20, 24, 22 players respectively. How many different ways can the worst team choose 3 players?

Observe, that the order of the players does not matter. So we are choosing 3 players from a pool of 66:

$$C(66,3) = \binom{66}{3} = \frac{66!}{63! \cdot 3!} = (66 \cdot 65 \cdot 64)/3! = 45,760$$

Combinations

How many ways are there to select (choose) x players from a 10-member tennis team to compete in an upcoming tournament, where $0 \leq x \leq 10$?

$$\begin{aligned}
x = 0: &\quad \tbinom{10}{0} = \tfrac{10!}{10! \cdot 0!} = 1 \\
x = 1: &\quad \tbinom{10}{1} = \tfrac{10!}{9! \cdot 1!} = 10 \\
x = 2: &\quad \tbinom{10}{2} = \tfrac{10!}{8! \cdot 2!} = 45 \\
x = 3: &\quad \tbinom{10}{3} = \tfrac{10!}{7! \cdot 3!} = 120 \\
x = 4: &\quad \tbinom{10}{4} = \tfrac{10!}{6! \cdot 4!} = 210 \\
x = 5: &\quad \tbinom{10}{5} = \tfrac{10!}{5! \cdot 5!} = 252 \\
x = 6: &\quad \tbinom{10}{6} = \tfrac{10!}{4! \cdot 6!} = 210 \\
x = 7: &\quad \tbinom{10}{7} = \tfrac{10!}{3! \cdot 7!} = 120 \\
x = 8: &\quad \tbinom{10}{8} = \tfrac{10!}{2! \cdot 8!} = 45 \\
x = 9: &\quad \tbinom{10}{9} = \tfrac{10!}{1! \cdot 9!} = 10 \\
x = 10: &\quad \tbinom{10}{10} = \tfrac{10!}{0! \cdot 10!} = 1
\end{aligned}$$

Do you see a pattern?

Combinations

Theorem: $C(n,r) = C(n,n-r)$

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} = C(n,r)$$

ALTERNATE PROOF:

Let S be a set with n elements. Every subset A with r elements corresponds uniquely to a subset of size $n-r$ elements (all those elements not in A).

Therefore the number of subsets of size r is the same as the number of subsets of size $n-r$: $C(n,r) = C(n,n-r)$.

The alternate proof above is an example of a **combinatorial proof**: it uses counting arguments, rather than algebraic manipultation.

Example 1

The english alphabet has 21 consonants and 5 vowels. How many strings of 6 letters contain:

(a) exactly one vowel?

(b) exactly two vowels?

(c) at least one vowel?

(d) at least two vowels?

Solution: First, we know there are 26^6 strings with length 6 and 21^6 strings with only consonants.

(a) There are 6 positions to place 1 vowel, and 5 ways to select the vowel. This leaves 5 free positions for consonants: $6 \cdot 5 \cdot 21^5$.

(b) There are $\binom{6}{2}$ ways to select the 2 positions for the vowels, and 5 ways to pick each vowel. This leaves 4 positions free for the consonants: $15 \cdot 5^2 \cdot 21^4$.

(c) Apply subtraction: the number of all strings minus the number with no vowel: $26^6 - 21^6$.

(d) Apply subtraction: (c) - (a) $= 26^6 - 21^6 - 6 \cdot 5 \cdot 21^5$.

Example 2

DNA sequences are often represented as strings over the alphabet AGCT. How many DNA sequences of length 10 have the following properties?

(a) There are exactly 3 G's?

(b) There are exactly 3 G's and 3 T's?

(c) There is a subsequence of the form TTT?

(d) There are exactly 3 G's or 3 A's?

(e) The first three letters are distinct.

Solution:

(a) The 3 G's can be placed in $\binom{10}{3}$ ways. The remaining 7 positions can be any of the other 3 letter. Thus: $\binom{10}{3} \cdot 3^7$.

(b) Place the G's first: $\binom{10}{3}$. Then place the T's: $\binom{7}{3}$. Fill the remaining 4 spots with A and C: 2^4. Thus: $\binom{10}{3}\binom{7}{3}2^4$.

(c) There must be at least 3 T's. Use subtraction: total strings minus strings with 0 T's, 1 T, and 2 T's: $4^{10} - 3^{10} - \binom{10}{1}3^9 - \binom{10}{2}3^8$.

(d) Must apply inclusion/exclusion. If we count all strings with 3 G's then add all strings with 3 A's, we will count strings with both 3 G's and 3 A's twice.

(e) $P(4, 3) \cdot 4^7$

Example 3

Given an ordering of 7 playing cards $c_1, c_2, \ldots c_7$, how many such orderings are there such that:

(a) The first 4 cards are all kings?

(b) The first 4 cards contain 4 of a kind?

(c) There exists 4 kings?

(d) There exists 4 of a kind?

(e) Answer (c) and (d) if the order of the cards does not matter?

(f) Why does (d) get more difficult if we deal a hand of 8 or more cards?

Solution:

(a) There are 4! ways to order the kings: $4! \cdot P(48, 3) = 4! \cdot 48 \cdot 47 \cdot 46$

(b) There are 13 choices for the 4 of a kind: $13 \cdot (a)$.

(c) Select spots for kings, then order the kings: $\binom{7}{4} \cdot (a)$.

(d) $13 \cdot (c)$

(e) (c): $\binom{48}{3}$ (d) $13 \cdot \binom{48}{3}$

(f) Must beware careful of double counting hands that have two four of a kind's. Count hands with exactly one four of a kind, then add hands with 2 four of a kinds.