Solving the Sigma-Tau Problem

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Abstract

Knuth assigned the following open problem a difficulty rating of 48/50 in *The Art of Computer Programming Volume 4A*:

> For odd $n \geq 3$, can the permutations of $\{1, 2, \ldots, n\}$ be ordered in a cyclic list so that each permutation is transformed into the next by applying either the operation σ, a rotation to the left, or τ, a transposition of the first two symbols?

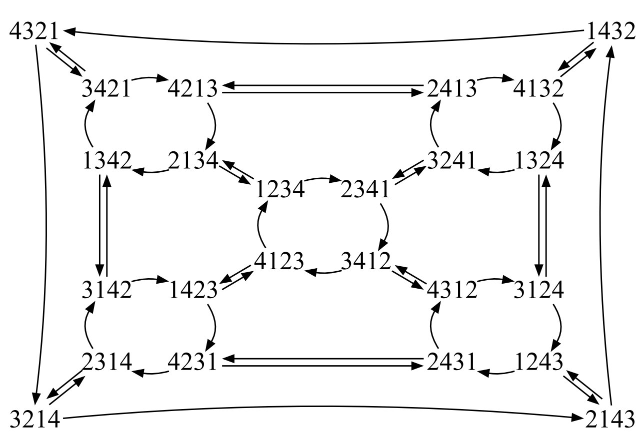
This problem, known as the *Sigma-Tau problem*, is equivalent to the problem of finding a Hamilton cycle on the directed Cayley graph generated by σ and τ. In this paper we solve the Sigma-Tau problem by providing a simple $O(n)$-time successor rule to generate successive permutations of a Hamilton cycle in the aforementioned Cayley graph.

1 Introduction

Let $\mathbf{P}_n$ denote the set of all permutations of $\{1, 2, \ldots, n\}$. Let $\pi = p_1 p_2 \cdots p_n$ be a permutation in $\mathbf{P}_n$ and consider the following two operations on π:

$$\sigma(\pi) = p_2 p_3 \cdots p_n p_1 \quad \text{and} \quad \tau(\pi) = p_2 p_1 p_3 p_4 \cdots p_n.$$

The operation σ rotates a permutation one position to the left and τ transposes the first two elements. The *Sigma-Tau graph* $\mathcal{G}_n$ is a directed graph where the vertices are the permutations $\mathbf{P}_n$. There is a directed edge from π_1 to π_2 if and only if $\pi_2 = \sigma(\pi_1)$ or $\pi_2 = \tau(\pi_1)$. Such a graph can be thought of as a Cayley graph over $\mathbf{P}_n$ with generators σ and τ. The Sigma-Tau graph $\mathcal{G}_4$ is illustrated below.



Sigma-Tau Problem Does there exist is a Hamilton cycle in $\mathcal{G}_n$ for odd $n \geq 3$?

This Sigma-Tau problem was assigned a difficulty of 48/50 in Knuth's *The Art of Computer Programming*, making it the hardest open problem in the fascicle version of Volume 4A [1, Problem 71 in Section 7.2.1.2] since the middle-levels problem which was rated 49/50 was recently solved by Mütze [2]. A reproduction of this question is shown below.

71. [*48*] Does the Cayley graph with generators $\sigma = (1\,2\,\ldots\,n)$ and $\tau = (1\,2)$ have a Hamiltonian cycle whenever $n \geq 3$ is odd?

From general Hamilton cycles conditions given by Rankin [4] (see also [8]), it is known that there is no Hamilton cycle in $\mathcal{G}_n$ for even $n > 2$. For $n = 3$, the following is a Hamilton cycle in $\mathcal{G}_3$:

$$231,\ 312,\ 132,\ 321,\ 213,\ 123.$$

It applies the operations $\sigma, \tau, \sigma, \sigma, \tau$ followed by σ to return to the first permutation. The Sigma-Tau problem can also be thought of as a combinatorial generation problem: *Can the permutations* $\mathbf{P}_n$ *be listed so that successive permutations (including the last/first) differ by the operation* σ *or* τ*?* The efficient ordering and generation of permutations has a long and interesting history with surveys by Sedgewick in the 1970s [7], Savage in the 1990s [5], and more recently by Knuth [1]. However the Sigma-Tau problem has remained a long-standing open problem in the area.

The Hamilton path variant of the Sigma-Tau problem was stated in 1975 in first edition of the *Combinatorial Algorithms* textbook by Nijenhuis and Wilf [3, Exercise 6]. An explicit Hamilton path in $\mathcal{G}_n$ was recently given by the authors in [6]. Many of the same concepts are revisited here to solve the significantly more difficult Hamilton cycle problem. Specifically, the main result of this paper is to answer the Sigma-Tau problem in the affirmative, providing a simple $O(n)$-time successor rule to produce successive permutations in a Hamilton cycle of $\mathcal{G}_n$.

In the following section, we present some necessary definitions and notation along with some preliminary results. In Section 3 we describe how $\mathcal{G}_n$ can be partitioned into 2 cycles, and then ultimately provide a construction for a Hamilton cycle in $\mathcal{G}_n$, for odd $n > 3$. The Appendix contains a C implementation for our Hamilton cycle construction. The construction presented in this article also appears in an unpublished manuscript [9] that provides an alternate proof using rotation systems.

2 Preliminary Definitions, Notation, and Results

Unless otherwise stated, assume for the rest of this paper that $n > 3$. Let $\pi = p_1p_2\cdots p_n$ denote a permutation in $\mathbf{P}_n$. Let $\mathbf{Q}$ be a subset of $\mathbf{P}_n$ that is closed under σ. A *successor rule* on $\mathbf{Q}$ is a function $f : \mathbf{Q} \rightarrow \mathbf{Q}$ that maps each permutation π to one of $\sigma(\pi)$ or $\tau(\pi)$. Our goal is to define a successor rule on $\mathbf{P}_n$, with the appropriate *conditions*, that constructs a Hamilton cycle one vertex (permutation) at a time in the Sigma-Tau graph $\mathcal{G}_n$. A template for the function is as follows:

$$f(\pi) = \begin{cases} \tau(\pi) & \text{if } \textit{conditions}; \\ \sigma(\pi) & \text{otherwise.} \end{cases}$$

Observe that the successor rule $f(\pi) = \sigma(\pi)$ partitions $\mathcal{G}_n$ into $(n{-}1)!$ cycles which correspond to equivalence classes of permutations under rotation. Let the lexicographically largest permutation in each cycle be its representative, and call such a permutation a *cyclic permutation*; each representative corresponds to a permutation starting with n. Let $rotations(\pi)$ denote the set of permutations rotationally equivalent to π.

Remark 2.1 *If a successor rule f induces a Hamilton cycle in $\mathcal{G}_n$ then there are at least $(n{-}1)!$ permutations π such that $f(\pi) = \tau(\pi)$.*

When representing a permutation, the last symbol can be inferred from the first $n{-}1$ symbols. A *shorthand permutation* is a length $n{-}1$ prefix of some permutation. For $1 \leq j \leq n{-}2$, define $g(j) = j{+}1$, and define $g(n{-}1) = 2$. A *seed* is a shorthand permutation $\boldsymbol{s} = s_1 s_2 \cdots s_{n-1}$ where $s_1 = n$ and the missing symbol x is $g(s_2)$ (Note: this definition is different from the one given in [6] and it is critical to our Hamilton cycle construction). Let $\boldsymbol{Seeds}_n$ denote the set of all $(n{-}1)(n{-}3)!$ seeds. Given a seed $\boldsymbol{s}$ with missing symbol x, the *flower of $\boldsymbol{s}$*, denoted by $\mathit{flower}(\boldsymbol{s})$, is the set of all $n{-}1$ cyclic permutations that can be obtained by inserting x after a symbol in $\boldsymbol{s}$. Given a seed $\boldsymbol{s}$, let $perms(\boldsymbol{s}) = \bigcup_{\pi \in \mathit{flower}(\boldsymbol{s})} rotations(\pi)$. If $\boldsymbol{S}$ is a set of seeds, let $perms(\boldsymbol{S}) = \bigcup_{\boldsymbol{s} \in \boldsymbol{S}} perms(\boldsymbol{s})$.

Example 1 When $n = 5$ the $4 \cdot 2! = 8$ seeds are:

$$5134,\ 5143,\ 5214,\ 5241,\ 5312,\ 5321,\ 5413,\ 5431.$$

The flower of seed 5321 is $\mathit{flower}(5321) = \{54321,\ 53421,\ 53241,\ 53214\}$.

$$\begin{aligned} perms(5321) \;=\; & \mathbf{54321}, 43215, 32154, 21543, 15432, \\ & \mathbf{53421}, 34215, 42153, 21534, 15342, \\ & \mathbf{53241}, 32415, 24153, 41532, 15324, \\ & \mathbf{53214}, 32145, 21453, 14532, 45321. \end{aligned}$$

Remark 2.2 *Every cyclic permutation $\pi = p_1 p_2 \cdots p_n$ belongs to the flower of either one or two seeds. It belongs to the flower of the seed obtained by removing $g(p_2)$ from π. Also if $p_2 = g(p_3)$, then it belongs to the flower of the seed obtained by removing p_2 from π.*

An immediate consequence is the following remark.

Remark 2.3 $perms(\boldsymbol{Seeds}_n) = \mathbf{P}_n$.

Our definitions of seeds and flowers are motivated by the following equivalence property. Given a permutation $\pi = p_1 p_2 \cdots p_n$, let $equiv(\pi)$ be the set of all rotations of $p_1 p_3 p_4 \cdots p_n$ with p_2 inserted back into the second position. For example $equiv(54321) = \{54321, 34215, 24153, 14532\}$. A successor rule f is *τ-equivalent* if $f(\pi) = \tau(\pi)$ implies that $f(\pi') = \tau(\pi')$ for all permutations $\pi' \in equiv(\pi)$.

Lemma 2.4 *A successor rule f induces a cycle cover on $\mathcal{G}_n$ if and only if f is τ-equivalent.*

Proof. (⇒) Suppose f induces a cycle cover on $\mathcal{G}_n$. If $f(\pi) = \tau(\pi)$ for some permutation $\pi = p_1p_2\cdots p_n$, then $\sigma(\pi) = p_2p_3\cdots p_np_1$ must be preceded by $\pi' = \tau(p_2p_3\cdots p_np_1) = p_3p_2p_4p_5\cdots p_np_1$. Thus, $f(\pi') = \tau(\pi')$. Repeating this argument starting with π' implies that $f(p_4p_2p_5p_6\cdots p_np_1p_3) = \tau(p_4p_2p_5p_6\cdots p_np_1p_3)$ and so on, which implies that f is τ-equivalent. (⇐) Suppose f is τ-equivalent. Consider $\pi = p_1p_2\cdots p_n$ and $\pi_1 = p_2p_1p_3p_4\cdots p_n$ and $\pi_2 = p_np_1p_2\cdots p_{n-1}$. Note that $\tau(\pi_1) = \sigma(\pi_2) = \pi$. For f to be a cycle cover on $\mathcal{G}_n$ exactly one of $f(\pi_1)$ and $f(\pi_2)$ must be π. This follows since $\pi_2 \in \mathit{equiv}(\pi_1)$. □

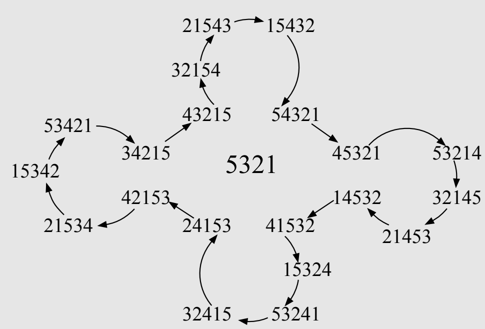
2.1 A Hamilton Cycle for an Induced Subgraph of $\mathcal{G}_n$

Let $\mathcal{G}_n[\mathbf{Q}]$ denote the subgraph of $\mathcal{G}_n$ induced by $\mathbf{Q}$. By considering the τ-equivalence property and considering a seed $\boldsymbol{s} = s_1s_2\cdots s_{n-1}$ with missing symbol x, we define a successor rule on $\mathcal{G}_n[\mathit{perms}(\boldsymbol{s})]$ that induces a Hamilton cycle. For $1 \leq j \leq n{-}1$, consider the cyclic permutation obtained by inserting x after s_j. Let π_j denote the rotation of this permutation such that x is in the second position. Define a τ-equivalent successor rule $f_{\boldsymbol{s}}$ on $\mathcal{G}_n[\mathit{perms}(\boldsymbol{s})]$ as follows:

$$f_{\boldsymbol{s}}(\pi) = \begin{cases} \tau(\pi) & \text{if } \pi = \pi_j \text{ for some } 1 \leq j \leq n{-}1; \\ \sigma(\pi) & \text{otherwise.} \end{cases}$$

Example 2 Consider seed $\boldsymbol{s} = 5321$ with missing symbol $x = 4$. Repeated application of the successor rule $f_{\boldsymbol{s}}$ induces the following Hamilton cycle in $\mathcal{G}_5[\mathit{perms}(5321)]$:

45321, 53214, 32145, 21453,
14532 $= \pi_4$,
41532, 15324, 53241, 32415,
24153 $= \pi_3$,
42153, 21534, 15342, 53421,
34215 $= \pi_2$,
43215, 32154, 21543, 15432,
54321 $= \pi_1$.



The five permutations in each row are equivalent under rotation. A τ transition is applied to move between the equivalence classes when the second symbol is the missing symbol $x = 4$.

Remark 2.5 $f_{\boldsymbol{s}}(\pi_j) = \tau(\pi_j) = \sigma(\pi_{j-1})$, *where* $\pi_0 = \pi_{n-1}$.

Let $\mathit{seq}(\pi)$ denote the following sequence of all permutations rotationally equivalent to π:

$$\sigma(\pi), \sigma^2(\pi), \ldots, \sigma^{n-1}(\pi), \pi,$$

where σ^j denotes $\sigma^{j-1}(\sigma(j))$ for $j > 1$. Repeated application of $f_{\boldsymbol{s}}$ induces a Hamilton cycle, denoted by $\mathit{ham}(\boldsymbol{s})$, in $\mathcal{G}_n[\mathit{perms}(\boldsymbol{s})]$ as follows:

$$\mathit{ham}(\boldsymbol{s}) = \mathit{seq}(\pi_{n-1}), \mathit{seq}(\pi_{n-2}), \ldots, \mathit{seq}(\pi_1).$$

Lemma 2.6 *For any seed* $\boldsymbol{s}$*, the successor rule* $f_{\boldsymbol{s}}$ *induces a Hamilton cycle in* $\mathcal{G}_n[\mathit{perms}(\boldsymbol{s})]$ *using* $n{-}1$ τ*-edges.*

2.2 A Tree-like Structure of Seeds

The seeds of the set $\boldsymbol{Seeds}_n$ can be arranged into a tree-like structure that has exactly one cycle. Consider a seed $\boldsymbol{s} = s_1s_2\cdots s_{n-1}$ with missing symbol x. Define the *parent* of $\boldsymbol{s}$, denoted by $parent(\boldsymbol{s})$, to be the seed obtained by removing $g(x)$ from $s_1xs_2\cdots s_{n-1}$. Let $\alpha(\boldsymbol{s})$ be the length $n{-}3$ prefix of $s_2(s_2{-}1)\cdots 2(n{-}1)(n{-}2)\cdots 2$. By this definition, the last element of $\alpha(\boldsymbol{s})$ is $g(x)$. The *decreasing subsequence* of $\boldsymbol{s}$ is the longest prefix of $\alpha(\boldsymbol{s})$ that appears as a subsequence in $s_3, s_4, \ldots, s_{j-1}$, where j is such that $s_j = 1$. This is well-defined since 1 appears in every seed, but not in the first position. The *level* of $\boldsymbol{s}$ is $(n{-}3)$ minus the length of its decreasing subsequence.

Example 3 The decreasing subsequence of the following seeds is highlighted in blue.

seed $\boldsymbol{s}$	$\alpha(\boldsymbol{s})$	**level**	$parent(\boldsymbol{s})$
64213	432	2	65413
63521	325	1	64321
64321	432	0	65431

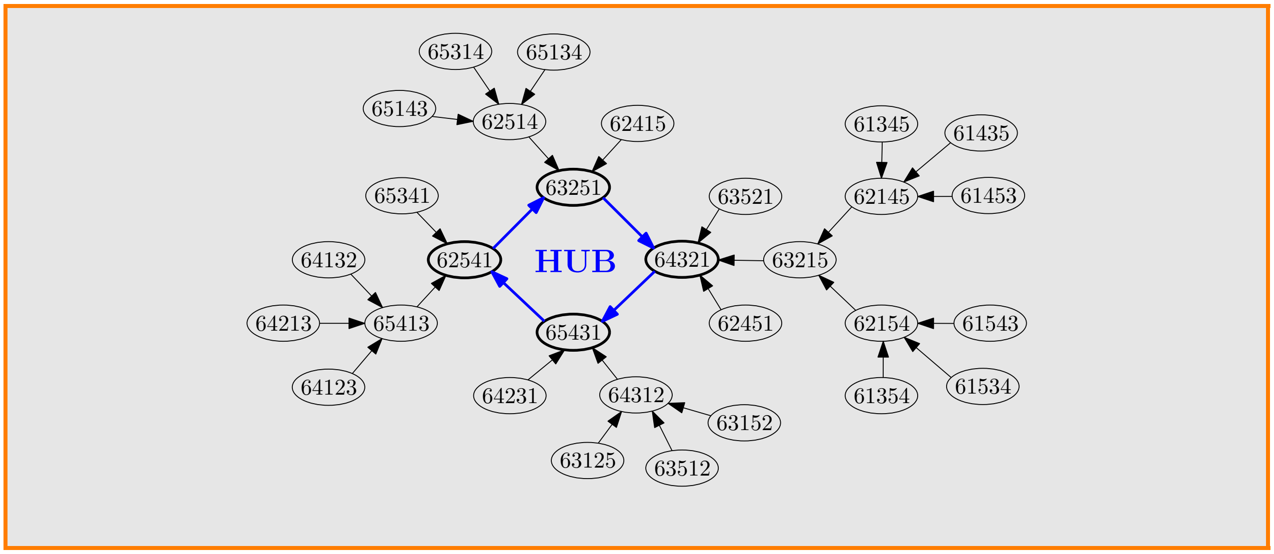
Lemma 2.7 *If $\boldsymbol{s}$ is a seed at level $\ell > 0$, then $parent(s)$ is at level $\ell{-}1$.*

Proof. Let $\boldsymbol{s} = s_1s_2\cdots s_{n-1}$ be a seed with missing symbol x. Since $\ell > 0$, the last symbol of $\alpha(\boldsymbol{s})$, which is $g(x)$, will not be in $\boldsymbol{s}$'s decreasing subsequence. Thus, the decreasing subsequence of $parent(\boldsymbol{s})$ is the decreasing subsequence of $\boldsymbol{s}$ with $g(x)$ added to the front. Thus, $parent(\boldsymbol{s})$ is at level $\ell{-}1$. □

Let $\boldsymbol{Hub}_n$ denote the subset of seeds at level 0. A seed $s_1s_2\cdots s_{n-1}$ with missing symbol x is in $\boldsymbol{Hub}_n$ if and only if $xs_2s_3\cdots s_{n-2}$ is a rotation of $(n{-}1)(n{-}2)\cdots 2$, $s_1 = n$, and $s_{n-1} = 1$. Denote the $n{-}2$ seeds in the $\boldsymbol{Hub}_n$ by $\boldsymbol{h}_1, \boldsymbol{h}_2, \ldots, \boldsymbol{h}_{n-2}$. They can be ordered as follows, where $parent(\boldsymbol{h}_j) = \boldsymbol{h}_{j+1}$ (with $\boldsymbol{h}_{n-1} = \boldsymbol{h}_1$) and each $\boldsymbol{h}_i$ is missing the symbol $i+1$.

$$\begin{array}{rcl} \boldsymbol{h}_1 & = & n(n{-}1)(n{-}2)\cdots 31, \\ \boldsymbol{h}_2 & = & n2(n{-}1)(n{-}2)\cdots 41, \\ \boldsymbol{h}_3 & = & n32(n{-}1)(n{-}2)\cdots 51, \\ \cdots & \cdots & \cdots \\ \boldsymbol{h}_{n-2} & = & n(n{-}2)(n{-}3)\cdots 1. \end{array}$$

Example 4 For $n = 6$, the parent structure of all seeds is illustrated below, where $\boldsymbol{h}_1 = 65431$, $\boldsymbol{h}_2 = 62541$, $\boldsymbol{h}_3 = 63251$, $\boldsymbol{h}_4 = 64321$.



Lemma 2.8 *Let $n > 4$ and let $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ be distinct seeds where $\boldsymbol{s}_1 = s_1 s_2 \cdots s_{n-1}$ has missing symbol x. If $\boldsymbol{s}_2 = parent(\boldsymbol{s}_1)$ then $flower(\boldsymbol{s}_1) \cap flower(\boldsymbol{s}_2) = \{s_1 x s_2 \cdots s_{n-1}\}$. If $\boldsymbol{s}_2 \neq parent(\boldsymbol{s}_1)$ and $\boldsymbol{s}_1 \neq parent(\boldsymbol{s}_2)$ then $flower(\boldsymbol{s}_1) \cap flower(\boldsymbol{s}_2) = \emptyset$.*

Proof. Suppose $\boldsymbol{s}_2 = parent(\boldsymbol{s}_1)$. From the definition of parent, $s_1 x s_2 \cdots s_{n-1}$ is in $flower(\boldsymbol{s}_1) \cap flower(\boldsymbol{s}_2)$. Every other cyclic permutation in $flower(\boldsymbol{s}_1)$ starts with $s_1 s_2$, where $s_2 = x{-}1$ or $s_2 = n{-}1$ and $x = 2$. Therefore since $n > 4$, these permutations are not in $flower(\boldsymbol{s}_2)$. Thus $flower(\boldsymbol{s}_1) \cap flower(\boldsymbol{s}_2) = \{s_1 x s_2 \cdots s_{n-1}\}$. Now suppose that $\boldsymbol{s}_2 \neq parent(\boldsymbol{s}_1)$ and $\boldsymbol{s}_1 \neq parent(\boldsymbol{s}_2)$ and $flower(\boldsymbol{s}_1) \cap flower(\boldsymbol{s}_2) \neq \emptyset$. Then $flower(\boldsymbol{s}_1) \cap flower(\boldsymbol{s}_2)$ must contain some cyclic permutation $\pi = s_1 s_2 \cdots s_j x s_{j+1} \cdots s_{n-1}$ where $2 \leq j \leq n{-}1$. Note that if $j = 1$ then $\boldsymbol{s}_2 = parent(\boldsymbol{s}_1)$. By removing any symbol from π except x or s_2, the resulting shorthand permutation is not seed, by its definition. However, if removing s_2 is a seed, then $\boldsymbol{s}_1 = parent(\boldsymbol{s}_2)$, a contradiction. Thus in this case $flower(\boldsymbol{s}_1) \cap flower(\boldsymbol{s}_2) = \emptyset$. □

This lemma along with the definition of $f_{\boldsymbol{s}}$ implies that given a seed $\boldsymbol{s} = s_1 s_2 \cdots s_{n-1}$ with missing symbol x, $s_1 x s_2 \cdots s_{n-1}$ is the unique permutation π in $perms(\boldsymbol{s}) \cap perms(parent(\boldsymbol{s}))$ such that $f_{\boldsymbol{s}}(\pi) = \tau(\pi)$. Let $\tau parent(\boldsymbol{s})$ denote this permutation $s_1 x s_2 \cdots s_{n-1}$.

3 Successor Rules to Construct Hamilton Paths/Cycles in $\mathcal{G}_n$

In this section, we start by showing that the following successor rule partitions $\mathcal{G}_n$ into two cycles. Then by modifying the rule for a single permutation, a successor rule is presented that constructs a Hamilton path in $\mathcal{G}_n$. By modifying the rule for $n{-}1$ permutations we obtain a successor rule that constructs a Hamilton cycle in $\mathcal{G}_n$ for odd n.

Let $\boldsymbol{S}$ be a subset of $\boldsymbol{Seeds}_n$. Define the successor rule $F_{\boldsymbol{S}}$ on $\mathcal{G}_n[perms(\boldsymbol{S})]$ as follows:

$$F_{\boldsymbol{S}}(\pi) = \begin{cases} \tau(\pi) & \text{if there exists } \boldsymbol{s} \in \boldsymbol{S} \text{ such that } \pi \in perms(\boldsymbol{s}) \text{ and } f_{\boldsymbol{s}}(\pi) = \tau(\pi); \\ \sigma(\pi) & \text{otherwise.} \end{cases}$$

Remark 3.1 *The successor rule $F_{\mathbf{S}}$ is τ-equivalent.*

As a first step, we focus on how this successor rule behaves on $\boldsymbol{Hub}_n$. For our upcoming Hamilton cycle construction on $\mathcal{G}_n$, we will want to keep track of some special permutations. Consider the $n{-}2$ permutations obtained by taking all rotations of $(n{-}1)\cdots 32$ and inserting n into the first position and 1 into the second last position:

$$\begin{array}{lll} \mathbf{p}_1 & = & n(n{-}2)\cdots 321(n{-}1), \\ \mathbf{p}_2 & = & n(n{-}3)\cdots 32(n{-}1)1(n{-}2), \\ \mathbf{p}_3 & = & n(n{-}4)\cdots 32(n{-}1)(n{-}2)1(n{-}3), \\ \cdots & \cdots & \cdots \\ \mathbf{p}_{n-2} & = & n(n{-}1)\cdots 4312. \end{array}$$

Define $\mathbf{p}_{n-1}$ as follows:

$$\mathbf{p}_{n-1} = n(n{-}3)(n{-}4)\cdots 2(n{-}2)(n{-}1)1.$$

Removing the second symbol from each of these $n{-}1$ permutations results in a seed at level 1 and each permutation is the τ*parent* of the resulting seed. The following example illustrates how $F_{\boldsymbol{Hub}_n}$ partitions $\mathcal{G}_n[\mathit{perms}(\boldsymbol{Hub}_n)]$ into two cycles for $n = 6$.

Example 5 For $n = 6$, $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_5$ are:

$$\mathbf{p}_1 = 643215,\ \mathbf{p}_2 = 632514,\ \mathbf{p}_3 = 625413,\ \mathbf{p}_4 = 654312,\ \mathbf{p}_5 = 632451.$$

$F_{\boldsymbol{Hub}_6}$ partitions $\mathcal{G}_6[\mathit{perms}(\boldsymbol{Hub}_6)]$ into the following two cycles C_1 and C_2. The cycle C_1 contains the permutations $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$ in that relative order highlighted in blue. The cycle C_2 contains $\mathbf{p}_5$ highlighted in blue.

$C_1 =$

564321, 643215, 432156, 321564, 215643, 156432, 516432, 164325, 643251,
463251, 632514, 325146, 251463, 514632, 146325, 416325, 163254, 632541,
362541, 625413, 254136, 541362, 413625, 136254, 316254, 162543, 625431,
265431, 654312, 543126, 431265, 312654, 126543, 216543, 165432, 654321.

$C_2 =$

543216,432165,321654, 231654,316542,165423,654231,542316,423165, 243165,431652,316524,165243,652431,524316,
254316,543162,431625, 341625,416253,162534,625341,253416,534162, 354162,541623,416235,162354,623541,235416,
325416,254163,541632, 451632,516324,163245,632451,324516,245163, 425163,251634,516342,163425,634251,342516,
432516,325164,251643, 521643,216435,164352,643521,435216,352164, 532164,321645,216453,164532,645321,453216.

Observe that C_1 starts with $\tau(654321)$ and ends with 654321 while C_2 begins with $\sigma(654321)$.

Lemma 3.2 *$F_{\boldsymbol{Hub}_n}$ partitions $\mathcal{G}_n[\mathit{perms}(\boldsymbol{Hub}_n)]$ into two cycles C_1 and C_2 where C_1 contains the permutations $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{n-2}$ while respecting their relative order, and C_2 contains $\mathbf{p}_{n-1}$. Moreover, C_1 contains $n\cdots 321$ and C_2 contains $(n{-}2)(n{-}1)(n{-}3)(n{-}4)\cdots 1n$.*

Proof. Since $F_{\boldsymbol{Hub}_n}$ is τ-equivalent, from Lemma 2.4 it will induce a cycle cover on $\mathcal{G}_n[perms(\boldsymbol{Hub}_n)]$. We explicitly show that it induces a two cycle cover with the properties mentioned. Given a $\boldsymbol{Hub}_n$ seed $\boldsymbol{h}_i = s_1s_2\cdots s_{n-1}$ with missing symbol $x = i+1$, define π_j^i in a similar manner used when defining π_j in $ham(\boldsymbol{s})$: it is the permutation obtained by inserting x after s_j in the seed $\boldsymbol{h}_i$, followed by a rotation so that x is in the second position. Let $\pi_j^0 = \pi_j^{n-2}$ and let $\pi_j^{n-1} = \pi_j^1$. Since $\boldsymbol{h}_i = n(i)(i{-}1)\cdots 2(n{-}1)(n{-}2)\cdots(i{+}2)1$,

$$\pi_{n-2}^i = (i{+}2)(i{+}1)1n(i)(i{-}1)\cdots 2(n{-}1)(n{-}2)\cdots(i{+}3).$$

Applying three rotations we have:

$$\sigma^3(\pi_{n-2}^i) = n(i)(i{-}1)\cdots 2(n{-}1)(n{-}2)\cdots(i{+}1)1 = \pi_1^{i-1}.$$

Now, from the definition of $ham(\boldsymbol{s})$ and Remark 2.5 we have

- $F_{\boldsymbol{Hub}_n}(\pi_1^{i-1}) = \tau(\pi_1^{i-1}) = \sigma(\pi_{n-1}^{i-1})$ which is the first permutation of $seq(\pi_{n-1}^{i-1})$,
- $F_{\boldsymbol{Hub}_n}(\pi_{n-1}^i) = \tau(\pi_{n-1}^i) = \sigma(\pi_{n-2}^i)$, and
- $F_{\boldsymbol{Hub}_n}(\pi_2^i) = \tau(\pi_2^i) = \sigma(\pi_1^i) = \sigma(\sigma^3(\pi_{n-2}^{i+1}))$.

Using these properties, we can explicitly trace the two cycles in $\mathcal{G}_n[perms(\boldsymbol{Hub}_n)]$. Let C_1 be the following cycle obtained by applying $F_{\boldsymbol{Hub}_n}$ starting from the first permutation of $seq(\pi_{n-1}^{n-2})$:

$$\begin{array}{cccc}
seq(\pi_{n-1}^{n-2}), & \sigma(\pi_{n-2}^{n-2}), & \sigma^2(\pi_{n-2}^{n-2}), & \sigma^3(\pi_{n-2}^{n-2}), \\
seq(\pi_{n-1}^{n-3}), & \sigma(\pi_{n-2}^{n-3}), & \sigma^2(\pi_{n-2}^{n-3}), & \sigma^3(\pi_{n-2}^{n-3}), \\
seq(\pi_{n-1}^{n-4}), & \sigma(\pi_{n-2}^{n-4}), & \sigma^2(\pi_{n-2}^{n-4}), & \sigma^3(\pi_{n-2}^{n-4}), \\
 & \cdots & & \\
seq(\pi_{n-1}^{1}), & \sigma(\pi_{n-2}^{1}), & \sigma^2(\pi_{n-2}^{1}), & \sigma^3(\pi_{n-2}^{1}).
\end{array}$$

The cycle C_1 contains $(n{+}3)(n{-}2)$ permutations. Each row corresponds to the first $n{+}3$ permutations for some $ham(\boldsymbol{h}_i)$. Also observe that for $1 \leq i \leq n{-}2$, $\mathbf{p}_i$ is a member of $rotations(\pi_{n-1}^{n-1-i})$. Thus $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{n-2}$ appear in C_1 respecting the relative order. Moreover, $\sigma^3(\pi_{n-2}^1) = \pi_1^{n-2} = n\cdots 321$ is the last permutation in C_1. Let C_2 be the following cycle obtained by applying $F_{\boldsymbol{Hub}_n}$ starting from $\sigma^4(\pi_{n-2}^1)$:

$$\begin{array}{ll}
\sigma^4(\pi_{n-2}^1),\ \sigma^5(\pi_{n-2}^1),\ \ldots, \sigma^n(\pi_{n-2}^1), & seq(\pi_{n-3}^1),\ seq(\pi_{n-4}^1),\ \ldots, seq(\pi_2^1), \\
\sigma^4(\pi_{n-2}^2),\ \sigma^5(\pi_{n-2}^2),\ \ldots, \sigma^n(\pi_{n-2}^2), & seq(\pi_{n-3}^2),\ seq(\pi_{n-4}^2),\ \ldots, seq(\pi_2^2), \\
\sigma^4(\pi_{n-2}^3),\ \sigma^5(\pi_{n-2}^3),\ \ldots, \sigma^n(\pi_{n-2}^3), & seq(\pi_{n-3}^3),\ seq(\pi_{n-4}^3),\ \ldots, seq(\pi_2^3), \\
\cdots & \cdots \\
\sigma^4(\pi_{n-2}^{n-2}),\ \sigma^5(\pi_{n-2}^{n-2}),\ \ldots, \sigma^n(\pi_{n-2}^{n-2}), & seq(\pi_{n-3}^{n-2}),\ seq(\pi_{n-4}^{n-2}),\ \ldots, seq(\pi_2^{n-2}).
\end{array}$$

The cycle C_2 contains the remaining $((n{-}3) + n(n-4))(n{-}2)$ permutations of $perms(\boldsymbol{Hub}_n)$. The permutation $\mathbf{p}_{n-1}$ belongs to $rotations(\pi_{n-3}^{n-3})$, and thus belongs to C_2. Moreover C_2 ends with $\pi_2^{n-2} = (n{-}2)(n{-}1)(n{-}3)(n{-}4)\cdots 1n$. □

Because of the tree-like structure of the seeds, we can treat the cycles C_1 and C_2 of $\boldsymbol{Hub}_n$ as a base case and then repeatedly add appropriate seeds to grow the two cycles.

Lemma 3.3 *Let $n > 4$ and let $\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_m$ be an increasing ordering of $\boldsymbol{Seeds}_n$ by level, where $m = (n{-}1)(n{-}3)!$. Let $\boldsymbol{S} = \{\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_j\}$ for some $n{-}2 \leq j \leq m$. Then $F_{\boldsymbol{S}}$ partitions $\mathcal{G}_n[perms(\boldsymbol{S})]$ into two cycles C_1 and C_2.*

Proof. The proof is by induction on j. The base case when $j = n{-}2$ is covered by Lemma 3.2 since the first $n{-}2$ seeds are the $\boldsymbol{Hub}_n$ seeds with level 0. Consider $\boldsymbol{S} = \{\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_j\}$ for $n{-}2 \leq j < m$. Inductively, assume that $F_{\boldsymbol{S}}$ partitions $\mathcal{G}_n[perms(\boldsymbol{S})]$ into two cycles C_1 and C_2. Since $F_{\{\boldsymbol{s}_{j+1}\}} = f_{\boldsymbol{s}_{j+1}}$, $F_{\{\boldsymbol{s}_{j+1}\}}$ induces a Hamilton cycle in $\mathcal{G}_n[perms(\boldsymbol{s}_{j+1})]$. By the ordering of the seeds, $\boldsymbol{s}_{j+1} = s_1 s_2 \cdots s_{n-1}$ has level $\ell > 0$ and all seeds at a smaller level are in $\{\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_j\}$. Thus, by Lemma 2.7 and Lemma 2.8 there is exactly one seed $\boldsymbol{s}$ in $\{\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_j\}$, namely $parent(\boldsymbol{s}_{j+1})$, such that $flower(\boldsymbol{s}_{j+1}) \cap flower(\boldsymbol{s})$ is not empty. Moreover this intersection contains the single cyclic permutation $\pi = s_1 x s_2 \cdots s_{n-1}$. Thus, from the definition of $ham(\boldsymbol{s}_j)$, π is the only permutation in $perms(\boldsymbol{S})$ such that $F_{\boldsymbol{S} \cup \{\boldsymbol{s}_{j+1}\}}(\pi)$ is not in $perms(\boldsymbol{S})$. Suppose that π is in C_1. By replacing the edge $(\pi, \sigma(\pi))$ in C_1 constructed by $F_{\boldsymbol{S}}$ from the inductive hypothesis with the sub-path of $ham(\boldsymbol{s}_{j+1})$ starting with π and ending with $\sigma(\pi)$, we obtain a larger cycle C_1 constructed by $F_{\boldsymbol{S} \cup \{\boldsymbol{s}_{j+1}\}}$ that contains all permutations in $perms(\boldsymbol{s}_{j+1})$. The case for when π is in C_2 is analogous. $\square$

When $\boldsymbol{S} = \boldsymbol{Seeds}_n$, the successor rule $F_{\boldsymbol{S}}$ is equivalent to the following.

2-cycle successor rule

Let $\pi = p_1 p_2 \cdots p_n$ be a permutation and let r be the symbol to the right of n when π is considered cyclically and skipping over p_2.

$$F(\pi) = \begin{cases} \tau(\pi) & \text{if } (r, p_2) \in \{(1,2), (2,3), \ldots, (n{-}2, n{-}1), (n{-}1, 2)\}; \\ \sigma(\pi) & \text{otherwise.} \end{cases}$$

3.1 Hamilton Path Successor

From Lemma 3.2, $F_{\boldsymbol{Hub}_n}$ partitions $\mathcal{G}_n[perms(\boldsymbol{Hub}_n)]$ into two cycles C_1 and C_2 where C_1 contains $\pi_1 = n \cdots 321$ and C_2 contains $\pi_2 = (n{-}2)(n{-}1)(n{-}3)(n{-}4) \cdots 1n$. Lemma 3.3 and its proof construction together with Remark 2.3 demonstrate that F partitions $\mathcal{G}_n$ into two cycles C_1 and C_2 where C_1 contains π_1 and C_2 contains π_2. Since $F(\pi_1) = \tau(\pi_1)$ and $F(\pi_2) = \tau(\pi_2)$ by changing the successor of π_1 from $\tau(\pi_1)$ to $\sigma(\pi_1) = \tau(\pi_2)$ in F we obtain a successor rule that constructs a Hamilton Path in $\mathcal{G}_n$ starting from $\tau(\pi_1)$ and ending with π_2.

Hamilton path successor rule for $\mathcal{G}_n$

Let $\pi = p_1 p_2 \cdots p_n$ be a permutation and let r be the symbol to the right of n when π is considered cyclically and skipping over p_2. Define the successor rule HP on $\mathcal{G}_n$ as follows:

$$HP(\pi) = \begin{cases} \tau(\pi) & \text{if } (r, p_2) \in \{(1,2), (2,3), \ldots, (n{-}2, n{-}1), (n{-}1, 2)\} \text{ and } \pi \neq n \cdots 321; \\ \sigma(\pi) & \text{otherwise.} \end{cases}$$

Our results prove the following theorem for $n > 4$. The correctness for cases $n = 2, 3, 4$ are easily verified by iterating $HP(\pi)$ starting from 12, 231, and 3421 respectively. For $n = 2$ we get 12, 21. For $n = 3$ we get 231, 312, 123, 213, 132, 321. For $n = 4$ we get:

3421, 4213, 2413, 4132, 1324, 3241, 2341, 3412, 4123, 1234, 2134, 1342,
3142, 1423, 4231, 2431, 4312, 3124, 1243, 2143, 1432, 4321, 3214, 2314.

Theorem 3.4 *The successor rule HP induces a Hamilton path in $\mathcal{G}_n$ starting from $\tau(n \cdots 321)$ and ending with $(n-2)(n-1)(n-3)(n-4)\cdots 1n$, for all $n > 1$.*

This Hamilton path successor is similar to, but not the same as the one presented in [6].

3.2 Hamilton Cycle Successor

To convert the 2-cycle successor F into a Hamilton cycle successor (which must be τ-equivalent by Lemma 2.4) we change the definition of $n-1$ transitions from σ to τ. Consider the $n-1$ permutations obtained by taking all rotations of $12\cdots(n-1)$ and inserting n into the second position:

$$\begin{array}{lcl}
\mathbf{r}_1 & = & (n-1)n12\cdots(n-2), \\
\mathbf{r}_2 & = & (n-2)n(n-1)12\cdots(n-3), \\
\mathbf{r}_3 & = & (n-3)n(n-2)(n-1)12\cdots(n-4), \\
\cdots & \cdots & \cdots \\
\mathbf{r}_{n-2} & = & 2n345\cdots(n-1)1, \\
\mathbf{r}_{n-1} & = & 1n23\cdots(n-1).
\end{array}$$

Let $\mathbf{R}_n = \{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{n-1}\}$. The following lemma is proved at the end of this subsection.

Lemma 3.5 *F partitions $\mathcal{G}_n$ into two cycles C_1 and C_2 where C_1 contains the permutations $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{n-2}$ while respecting their relative order, and C_2 contains $\mathbf{r}_{n-1}$.*

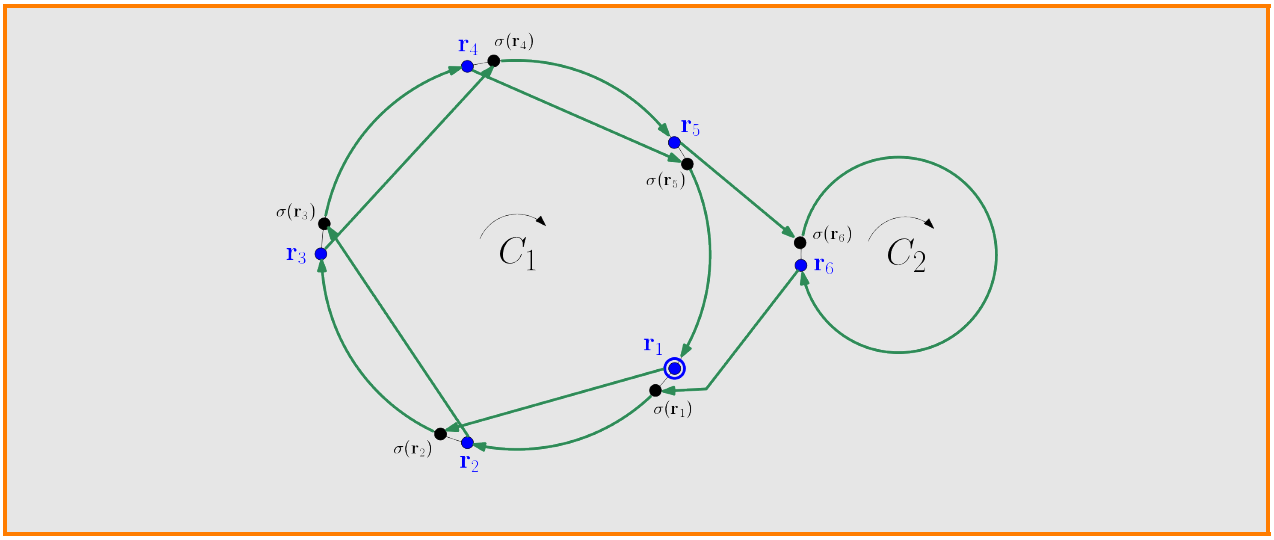
By changing the definition of F for the permutations in $\mathbf{R}_n$, we obtain the following successor rule.

Hamilton cycle successor rule for $\mathcal{G}_n$ where $n > 3$ is odd

Let $\pi = p_1p_2\cdots p_n$ be a permutation and let r be the symbol to the right of n when π is considered cyclically and skipping over p_2. Define the successor rule HC on $\mathcal{G}_n$ as follows:

$$HC(\pi) = \begin{cases} \tau(\pi) & \text{if } (r, p_2) \in \{(1,2),(2,3),\ldots,(n-2,n-1),(n-1,2)\} \text{ or } \pi \in \mathbf{R}_n; \\ \sigma(\pi) & \text{otherwise.} \end{cases}$$

Example 6 An illustration of how the successor rule $HC(\pi)$ joins the two cycles C_1 and C_2 constructed by applying the 2-cycle successor F on $\mathcal{G}_7$ is given below.



Theorem 3.6 *The successor rule HC induces a Hamilton cycle in $\mathcal{G}_n$, for odd $n > 3$.*

Proof. From Lemma 3.5, F partitions $\mathcal{G}_n$ into two cycles C_1 and C_2 where C_1 contains the permutations $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{n-2}$ while respecting their relative order, and C_2 contains $\mathbf{r}_{n-1}$. Observe that $\tau(\mathbf{r}_i) = \sigma(\mathbf{r}_{i+1})$ for $1 \leq i < n-1$ and $\tau(\mathbf{r}_{n-1}) = \sigma(\mathbf{r}_1)$. Also, $F(\mathbf{r}_i) = \sigma(\mathbf{r}_i)$ for all i. Considering C_1, let $\mathbf{q}_i$ denote the permutation before $\mathbf{r}_{i+1}$ for $1 \leq i < n-2$ and let $\mathbf{q}_{r-2}$ denote the permutation before $\mathbf{r}_1$. Then C_1 is given by

$$C_1 = \mathbf{r}_1, \sigma(\mathbf{r}_1), \ldots, \mathbf{q}_1, \quad \mathbf{r}_2, \sigma(\mathbf{r}_2), \ldots, \mathbf{q}_2, \quad \mathbf{r}_3, \sigma(\mathbf{r}_3), \ldots, \mathbf{q}_3, \quad \cdots \quad \mathbf{r}_{n-2}, \sigma(\mathbf{r}_{n-2}), \ldots, \mathbf{q}_{n-2}.$$

Similarly, letting $\mathbf{q}_{n-1}$ denote the permutation before $\mathbf{r}_{n-1}$ in C_2 we have

$$C_2 = \mathbf{r}_{n-1}, \sigma(\mathbf{r}_{n-1}), \ldots, \mathbf{q}_{n-1}.$$

By changing the successor of each $\mathbf{r}_i$ from $\sigma(\mathbf{r}_i)$ to $\tau(\mathbf{r}_i)$ in F we obtain HC which produces the following Hamilton cycle for odd n:

$$\mathbf{r}_1, \sigma(\mathbf{r}_2), \ldots, \mathbf{q}_2, \quad \mathbf{r}_3, \sigma(\mathbf{r}_4), \ldots, \mathbf{q}_4, \quad \cdots \quad \mathbf{r}_{n-2}, \sigma(\mathbf{r}_{n-1}), \ldots, \mathbf{q}_{n-1}, \quad \mathbf{r}_{n-1}, \sigma(\mathbf{r}_1), \ldots, \mathbf{q}_1,$$

$$\mathbf{r}_2, \sigma(\mathbf{r}_3), \ldots, \mathbf{q}_3, \quad \mathbf{r}_4, \sigma(\mathbf{r}_5), \ldots, \mathbf{q}_5, \quad \cdots \quad \mathbf{r}_{n-3}, \sigma(\mathbf{r}_{n-2}), \ldots, \mathbf{q}_{n-2}.$$

□

A complete C implementation of both the Hamilton path and Hamilton cycle successors is given in the Appendix.

3.2.1 Proof of Lemma 3.5

Recall that $F = F_{\boldsymbol{Seeds}_n}$. For each $\mathbf{r}_j$, $\pi = \sigma(\mathbf{r}_j) = p_1 p_2 \cdots p_n$ is a cyclic permutation where $p_2 \neq g(p_3)$. Thus, by Remark 2.2, π belongs exclusively to the flower of the seed obtained by removing $g(p_2)$ from π. Denote this seed by $sd(\mathbf{r}_j)$. Given a seed $\boldsymbol{s}$ at level $\ell > 0$, define $prehub(\boldsymbol{s})$ to be the seed at level 1 obtained by applying the parent operation $\ell - 1$ times starting with $\boldsymbol{s}$.

Lemma 3.7 *If $1 \leq j \leq n{-}2$ then $prehub(sd(\mathbf{r}_j))$ is the seed obtained by removing the first symbol of $\sigma^j((n{-}1)(n{-}2)\cdots 2)$, inserting n at the beginning and inserting 1 into the second last position. Additionally, $prehub(sd(\mathbf{r}_{n-1})) = n(n{-}4)(n{-}5)\cdots 2(n{-}2)(n{-}1)1$.*

Proof. The decreasing subsequence of $sd(\mathbf{r}_1) = n134\cdots(n{-}1)$ has length 0. Thus $\mathbf{r}_1$ is at level $n{-}3$. Applying $n{-}4$ parent operations we obtain the seed $n(n{-}3)(n{-}4)\cdots 21(n{-}1)$ at level 1, which is $prehub(sd(\mathbf{r}_1))$. For $2 \leq j \leq n{-}2$, consider $\mathbf{r}_j = (n{-}j)n(n{-}j{+}1)\cdots(n{-}1)12\cdots(n{-}j{-}1)$. The decreasing subsequence of $sd(\mathbf{r}_j)$ is simply $(n{-}j{+}1)$ with length 1. Thus, $n{-}5$ applications of the parent operation are required to get to $prehub(sd(\mathbf{r}_j))$ and this will yield the required seed. The decreasing subsequence of $sd(\mathbf{r}_{n-1}) = n245\cdots(n{-}1)1$ is $2(n{-}1)$, which has length 2. Applying $n{-}6$ parent operations we obtain the seed $n(n{-}4)(n{-}5)\cdots 2(n{-}2)(n{-}1)1$ at level 1, which is $prehub(sd(\mathbf{r}_{n-1}))$. □

By inserting the missing symbol from $prehub(sd(\mathbf{r}_j))$ into the second position we obtain $\mathbf{p}_j$.

Corollary 3.8 *For $1 \leq j \leq n{-}1$, the permutation $\tau parent(prehub(sd(\mathbf{r}_j))) = \mathbf{p}_j$.*

From Lemma 3.2, $F_{\boldsymbol{Hub}_n}$ partitions $\mathcal{G}_n[perms(\boldsymbol{Hub}_n)]$ into two cycles C_1 and C_2 where C_1 contains $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{n-2}$ in that relative order and C_2 contains $\mathbf{p}_{n-1}$. Lemma 3.3 and its proof construction, along with Remark 2.3 demonstrate that F partitions $\mathcal{G}_n$ into two cycles C_1 and C_2 where C_1 contains $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{n-2}$ in that relative order and C_2 contains $\mathbf{p}_{n-1}$. Together, Corollary 3.8, the inductive proof of Lemma 3.3, and the tree-like structure of the seeds imply that C_1, considered starting from $\mathbf{p}_1$, will be of the form:

$$\mathbf{p}_1, \ldots, \mathbf{r}_1, \ldots, \mathbf{p}_2, \ldots, \mathbf{r}_2, \ldots, \cdots, \mathbf{p}_{n-2}, \ldots, \mathbf{r}_{n-2}, \ldots$$

It also means that $\mathbf{r}_{n-1}$ is in C_2. This proves Lemma 3.5.

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Appendix - C code for a Hamilton path in $\mathcal{G}_n$ or a Hamilton cycle in $\mathcal{G}_n$ for odd n

```c
#include <stdio.h>
int n, pi[100], PATH=0, CYCLE=0;
//------------------------------------------------------
void Print() {
    for (int i=1; i<=n; i++) printf("%d", pi[i]); printf("\n");
}
//------------------------------------------------------
void Sigma() {
    int tmp, i;

    tmp = pi[1];
    for (i=1; i < n; i++) pi[i] = pi[i+1];
    pi[n] = tmp;
}
//------------------------------------------------------
void Tau() {
    int tmp = pi[1];  pi[1] = pi[2];  pi[2] = tmp;
}
//------------------------------------------------------
// RETURN TRUE IF pi[1..n]= n..21
int SpecialPerm() {
    for (int i=1; i<=n; i++) if (pi[i] != n-i+1) return 0;
    return 1;
}
//------------------------------------------------------
// RETURN TRUE IF pi[1]pi[3..n] is a rotation of 12..n-1
int SpecialSet() {

    if (pi[2] != n) return 0;
    if (pi[1] < n-1 &&  pi[1]+1 != pi[3]) return 0;
    if (pi[1] == n-1 && pi[3] != 1) return 0;
    for (int i=3; i<n; i++) {
        if (pi[i] < n-1 && pi[i]+1 != pi[i+1]) return 0;
        if (pi[i] == n-1 && pi[i+1] != 1) return 0;
    }
    return 1;
}
//------------------------------------------------------
void Next() {
    int r,i=1;

    while(pi[i] != n) i++;
    if (i == 1) r = pi[3];
    else if (i == n) r = pi[1];
    else r = pi[i+1];

    if (PATH && SpecialPerm()) Sigma();
    else if ((r < n-1 && pi[2]==r+1) || (r==n-1 && pi[2]==2)) Tau();
    else if (CYCLE && SpecialSet()) Tau();
    else Sigma();
}
//------------------------------------------------------
int main() {
    int total=0, TOTAL=1, i, type;

    printf("ENTER 1 (Hamilton Path) or 2 (Hamilton Cycle):"); scanf("%d", &type);
    if (type == 1) PATH = 1;
    if (type == 2) CYCLE = 1;
    printf("ENTER n (must be odd for cycle): "); scanf("%d", &n);

    for (i=2; i<=n; i++) TOTAL = TOTAL *i;       // TOTAL = n!
    for (i=1; i<=n; i++) pi[i] = n-i+1;          // INITAL PERM = tau(n..21)
    Tau();

    while (total < TOTAL) {
        Print();
        Next();
        total++;
}   }
```