# Non-impeding Noisy-AND Tree Causal Models Over Multi-valued Variables

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**Abstract**

To specify a Bayesian network (BN), a conditional probability table (CPT), often of an effect conditioned on its $n$ causes, must be assessed for each node. Its complexity is generally exponential in $n$. Noisy-OR and a number of extensions reduce the complexity to linear, but can only represent reinforcing causal interactions. Non-impeding noisy-AND (NIN-AND) trees are the first causal models that explicitly express reinforcement, undermining, and their mixture. Their acquisition has a linear complexity, in terms of both the number of parameters and the size of the tree topology. As originally proposed, however, they allow only binary effects and causes. This work generalizes binary NIN-AND tree models to multi-valued effects and causes. It is shown that the generalized NIN-AND tree models express reinforcement, undermining, and their mixture at multiple levels, relative to each active value of the effect. The model acquisition is still efficient. For binary variables, they degenerate into binary NIN-AND tree models. Hence, this contribution enables CPTs of discrete BNs of arbitrary variables (binary or multi-valued) to be specified efficiently through the intuitive concepts of reinforcement and undermining.

*Key words:* Bayesian networks, causal probabilistic models, conditional probability distributions, knowledge acquisition.

## 1. Introduction

To specify a BN, a CPT must be assessed for each non-root node. It is often advantageous to construct BNs along the causal direction, in which case a CPT is the distribution of an effect conditioned on its $n$ causes. In general, assessment of a CPT has complexity exponential in $n$. Noisy-OR [1] is the most well known technique that reduces this complexity to linear. A number of extensions have also been proposed such as [2, 3, 4]. However, noisy-OR, noisy-AND [3], as well as related techniques, can only represent causal interactions that are reinforcing [5].

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The NIN-AND tree [5] extends noisy-OR and provides the first causal model that explicitly expresses reinforcing and undermining causal interactions, as well as their mixtures at multiple levels. It requires specification of a set of probability parameters of a size linear in $n$, and a tree topology also of a size linear in $n$, which expresses the types of causal interactions among causes. The model uses default independence assumptions to gain efficiency, but is also flexible enough to allow these assumptions to be relaxed. By relaxing these assumptions incrementally and specifying more parameters accordingly, any CPT can be encoded through a NIN-AND tree.

As originally proposed [5], the effect and cause variables in a NIN-AND tree are binary, which limits its scope of applicability. In this work, we draw from the generalization of noisy-OR from the binary case, such as [6, 7], and generalize the NIN-AND tree model to multi-valued effects and cause variables.

The remainder of the paper is organized as follows: Section 2 reviews the binary NIN-AND tree models. We then introduce the terminology on graded multi-causal events in Section 3. The basic processing units in a NIN-AND tree model, NIN-AND gates, are generalized to graded multi-causal events in Section 4. This is followed by the definition of the generalized NIN-AND tree model in Section 6. Section 7 analyzes properties of these models in terms of expressiveness of reinforcement and undermining, as well as the complexity for acquiring a model.

## 2. Background on Binary NIN-AND Trees

This section is mostly based on [5]. An *uncertain cause* is a cause that can produce an effect but does not always do so. For instance, flu is a uncertain cause of fever. Denote a binary effect variable by $e$ and a set of binary cause variables of $e$ by $X = \{c_1, ..., c_n\}$. Denote $e = true$ by $e^+$ and $e = false$ by $e^-$. Similarly, for each cause $c_i$, denote $c_i = true$ by $c_i^+$ and $c_i = false$ by $c_i^-$.

A *causal success event* refers to an event that a cause $c_i$ caused an effect $e$ to occur successfully when all other causes of $e$ are inactive (equal to false). Denote this causal event by $e^+ \leftarrow c_i^+$ and its probability by $P(e^+ \leftarrow c_i^+)$. A *causal failure event* is an event where $e$ is false when $c_i$ is true and all other causes of $e$ are false. It is denoted by $e^- \leftarrow c_i^+$. Denote the causal event that a set $X = \{c_1, ..., c_n\}$ of causes caused $e$ by $e^+ \leftarrow c_1^+, ..., c_n^+$ or $e^+ \leftarrow \underline{x}^+$. Denote the set of *all causes* of $e$ by $C$.

The CPT $P(e|C)$ relates to probabilities of causal events as follows: If $C = \{c_1, c_2, c_3\}$, then $P(e^+|c_1^+, c_2^-, c_3^+) = P(e^+ \leftarrow c_1^+, c_3^+)$. Note that in our notation, a causal probability (the right-hand side) is always equivalent to a conditional probability (the left-hand side). However, an arbitrary conditional probability may not correspond to any causal probability. $C$ is assumed to include a leaky variable (if any) to capture causes that we do not wish to represent explicitly, and hence $P(e^+|c_1^-, c_2^-, c_3^-) = 0$.

Readers familiar with noisy-OR may draw the similarity (and difference) between the above and Pearl's formulation (Section 4.3.2 in [1]) of noisy-OR.

Pearl treats a cause as deterministic, whose occurrence always results in the effect, unless being blocked by an inhibitor. He encodes the causal uncertainty through the uncertain inhibitor. The conjunction of a deterministic cause and a stochastic inhibitor in his formulation is equivalent to the uncertain cause introduced above. Pearl depicts interaction among noisy-OR causes by a noisy-OR gate (Fig. 4.20 in [1]). The causal success event $e^+ \leftarrow \underline{x}^+$ defined above corresponds to the output event of a noisy-OR gate when it is true, and the causal failure event $e^- \leftarrow \underline{x}^+$ corresponds to the output event when it is false. Note that noisy-OR represents only reinforcing causal interaction, while our formulation is intended to express undermining in addition as well as their mixture, as shown below. Note also that computation of probability of the output event of a noisy-OR gate in terms of conditional probability in [1] is parallel to the relation between causal and conditional probabilities as illustrated above in $P(e^+|c_1^+, c_2^-, c_3^+) = P(e^+ \leftarrow c_1^+, c_3^+)$.

Causes reinforce each other if collectively they are at least as effective in causing the effect as some acting by themselves. If collectively they are less effective, then they undermine each other. For an example of reinforcement, consider curing of a type of cancer as the effect. Both radiotherapy and chemotherapy are its uncertain causes. When both therapies are applied, the chance of curing the cancer is improved. For an example of undermining, let the effect be the happiness of a person. Taking either one of two desirable jobs is a uncertain cause of the effect. When taking both, the chance of happiness is reduced due to overstress.

Note that if $C = \{c_1, c_2\}$ and $c_1$ and $c_2$ undermine each other, then all the following hold:

$$P(e^+|c_1^-, c_2^-) = 0, \quad P(e^+|c_1^+, c_2^-) > 0, \quad P(e^+|c_1^-, c_2^+) > 0,$$

$$P(e^+|c_1^+, c_2^+) < min(P(e^+|c_1^+, c_2^-), P(e^+|c_1^-, c_2^+)).$$

The following Def.1 defines the two types of causal interactions generally.

**Definition 1.** *Let $R = \{W_1, W_2, ...\}$ be a partition of a set $X$ of causes, $R' \subset R$ be any proper subset of $R$, and $Y = \cup_{W_i \in R'} W_i$. Sets of causes in $R$* **reinforce** *each other, iff*

$$\forall R' \ P(e^+ \leftarrow \underline{y}^+) \leq P(e^+ \leftarrow \underline{x}^+).$$

*Sets of causes in $R$* **undermine** *each other, iff*

$$\forall R' \ P(e^+ \leftarrow \underline{y}^+) > P(e^+ \leftarrow \underline{x}^+).$$

Note that reinforcement and undermining can occur between individual variables as well as sets of variables. When the causal interaction is between individual variables, each $W_i$ above is a singleton. Otherwise, each $W_i$ can be a generic set of causes. For instance, consider

$$X = \{c_1, c_2, c_3, c_4\}, W_1 = \{c_1, c_2\}, W_2 = \{c_3, c_4\}, R = \{W_1, W_2\}.$$

It is possible that causes $c_1$ and $c_2$ are reinforcing each other, and so are $c_3$ and $c_4$. But the sets $W_1$ and $W_2$ are undermining each other. A numerical example will be given below with Fig. 2.

Disjoint sets of causes $W_1, ..., W_m$ satisfy *failure conjunction* iff

$$(e^- \leftarrow \underline{w}_1^+, ..., \underline{w}_m^+) = (e^- \leftarrow \underline{w}_1^+) \wedge ... \wedge (e^- \leftarrow \underline{w}_m^+).$$

That is, collective failure is attributed to individual failures. They also satisfy *failure independence* iff

$$P((e^- \leftarrow \underline{w}_1^+) \wedge ... \wedge (e^- \leftarrow \underline{w}_m^+)) = P(e^- \leftarrow \underline{w}_1^+) \times ... \times P(e^- \leftarrow \underline{w}_m^+). \quad (1)$$

Disjoint sets of causes $W_1, ..., W_m$ satisfy *success conjunction* iff

$$(e^+ \leftarrow \underline{w}_1^+, ..., \underline{w}_m^+) = (e^+ \leftarrow \underline{w}_1^+) \wedge ... \wedge (e^+ \leftarrow \underline{w}_m^+).$$

That is, collective success requires individual effectiveness. They also satisfy *success independence* iff

$$P((e^+ \leftarrow \underline{w}_1^+) \wedge ... \wedge (e^+ \leftarrow \underline{w}_m^+)) = P(e^+ \leftarrow \underline{w}_1^+) \times ... \times P(e^+ \leftarrow \underline{w}_m^+). \quad (2)$$

It has been shown [5] that causes are undermining when they satisfy success conjunction and independence. Hence, undermining can be modeled by a direct NIN-AND gate as shown in the left of Fig. 1. Its root nodes (top) are causal

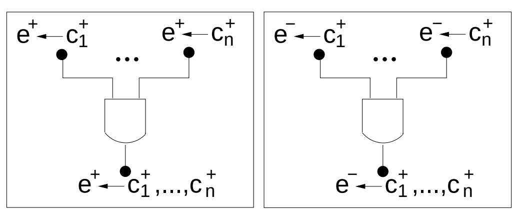


Figure 1: Direct (left) and dual (right) NIN-AND gates

success events of single causes, and its leaf node (bottom) is the causal success event in question. Success conjunction is expressed by the AND gate, and success independence is signified by disconnection of root nodes other than through the gate. The probability of the leaf event can be computed by Eqn. (2). Note that a direct NIN-AND gate differs from the common noisy-AND gate in that the former is non-impeding while the latter is impeding (see reference above for a detailed analysis).

It has also been shown [5] that causes are reinforcing when they satisfy failure conjunction and independence. Hence, reinforcement can be modeled by a dual NIN-AND gate (right). Its root nodes (top) are causal failure events of single causes, and its leaf node (bottom) is the causal failure event in question. Failure conjunction is expressed by the AND gate, and failure independence by disconnection of root nodes other than through the gate. The leaf event probability is computed by Eqn. (1). Note that the common noisy-OR gate is a special case of a dual NIN-AND gate in that the former allows only single cause input events while the latter allows multiple cause input events (see reference above for a formal analysis).

By using multiple direct and dual NIN-AND gates and organizing them into a tree topology, both reinforcement and undermining, as well as their mixture at multiple levels can be expressed in a single model, called a NIN-AND tree.

**Example 1.** *Consider an example where* $C = \{c_1, c_2, c_3\}$*,* $c_1$ *and* $c_3$ *undermine each other, but collectively they reinforce* $c_2$*. Assuming the default conjunction*

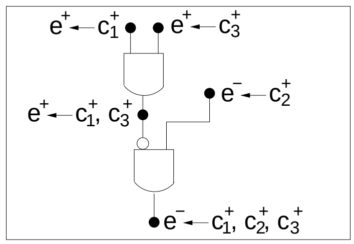


Figure 2: A NIN-AND tree causal model.

*and independence, their causal interaction (a two-level mixture of reinforcement and undermining), relative to the event* $e^- \leftarrow c_1^+, c_2^+, c_3^+$ *can be expressed by the NIN-AND tree shown in Fig. 2. It has five nodes (three root nodes at the top and one leaf node at the bottom) labeled by causal events. It has two NIN-AND gates. The top gate is direct and the bottom gate (the* leaf gate*) is dual. The link downward from node* $e^+ \leftarrow c_1^+, c_3^+$ *has a white circle end (a* negation *link) and negates the event. All other links are* forward *links that feeds the event in one end to the other.*

Given a NIN-AND tree, the probability of the leaf event can be computed by Algorithm 1.

**Algorithm 1.** *GetCausalEventProb(T)*
*Input: A NIN-AND tree* $T$ *of leaf node* $v$ *and leaf gate* $g$*, with probabilities of root events specified.*

*for each node* $w$ *directly inputting to* $g$*, do*
  *if* $P(w)$ *is not specified,*
    *denote the sub-NIN-AND-tree with* $w$ *as the leaf node by* $T_w$*;*
    $P(w) = GetCausalEventProb(T_w)$*;*
  *if* $(w, g)$ *is a forward link,* $P'(w) = P(w)$*;*
  *else* $P'(w) = 1 - P(w)$*;*
*return* $P(v) = \prod_w P'(w)$*;*

**Example 2.** *For the example in Fig. 2, after the following are specified,*

$$P(e^+ \leftarrow c_1^+) = 0.85,\ P(e^+ \leftarrow c_2^+) = 0.8,\ P(e^+ \leftarrow c_3^+) = 0.7,$$

*the probability* $P(e^- \leftarrow c_1^+, c_2^+, c_3^+) = 0.081$ *can be derived. From the above and other NIN-AND tree models derived from Fig. 2 (by removing some nodes, gates and links), the CPT in Table 1 can be derived.* $P(e^+|c_1^+, c_2^-, c_3^+)$ *is less*

Table 1: The CPT of an example NIN-AND tree model.

| | | | |
|---|---|---|---|
| $P(e^+\|c_1^-, c_2^-, c_3^-)$ | 0 | $P(e^+\|c_1^+, c_2^-, c_3^+)$ | 0.595 |
| $P(e^+\|c_1^+, c_2^-, c_3^-)$ | 0.85 | $P(e^+\|c_1^+, c_2^+, c_3^-)$ | 0.97 |
| $P(e^+\|c_1^-, c_2^+, c_3^-)$ | 0.8 | $P(e^+\|c_1^-, c_2^+, c_3^+)$ | 0.94 |
| $P(e^+\|c_1^-, c_2^-, c_3^+)$ | 0.7 | $P(e^+\|c_1^+, c_2^+, c_3^+)$ | 0.919 |

*than either* $P(e^+|c_1^+, c_2^-, c_3^-)$ *or* $P(e^+|c_1^-, c_2^-, c_3^+)$*: the result of undermining.* $P(e^+|c_1^+, c_2^+, c_3^+)$ *is larger than both* $P(e^+|c_1^+, c_2^-, c_3^+)$ *and* $P(e^+|c_1^-, c_2^+, c_3^-)$*: the result of reinforcement. A more efficient method to obtain the CPT from a NIN-AND tree model over* $C$ *can be found in [8].*

In general, for every cause $e$ and its set $C$ of causes, every NIN-AND tree, where no cause in $C$ appears in more than one root node, defines a CPT parameterized by the causal probabilities of its root events.

Before closing this section, we discuss the relation between NIN-AND tree models and the DeMorgan models [9]. Being unaware of the work reported

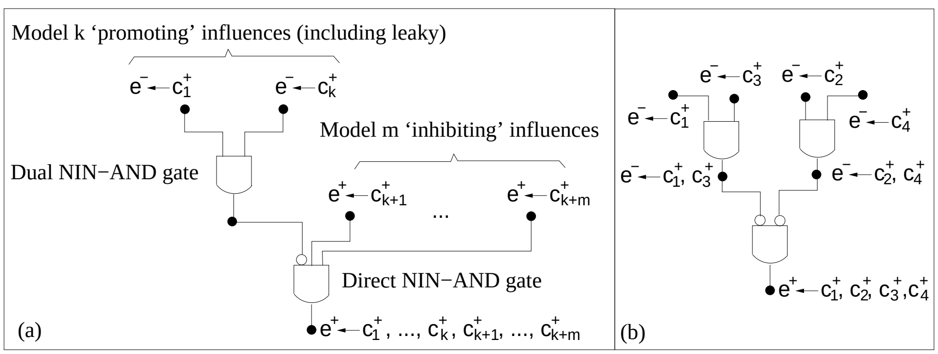


Figure 3: (a) The NIN-AND tree encoding of DeMorgan models. (b) A NIN-AND tree model not expressible as DeMorgan models.

in [10, 5] on NIN-AND tree models, authors of [9] independently developed DeMorgan models. Like binary NIN-AND trees, DeMorgan models deal with only binary effect and causes. The causal interaction mechanism of DeMorgan models is a special case of that of binary NIN-AND tree models, in the sense that every DeMorgan model can be encoded into a NIN-AND tree model of exactly two NIN-AND gates, as shown in Fig. 3 (a). On the other hand, NIN-AND tree models may contain three or more NIN-AND gates and thus may not be expressed as DeMorgan models. An example is shown in Fig. 3 (b), where causes $c_1$ and $c_3$ reinforce each other, so do $c_2$ and $c_4$, but the two groups undermine each other.

## 3. Graded Multi-Causal Events

Let $e$ be a multi-valued effect variable whose finite domain is denoted $D_e = \{e^0, e^1, ..., e^\eta\}$, where $\eta \geq 1$. The value $e^0$ (by the superscript index 0) represents the absence of the effect condition. Each value $e^j$ with a higher superscript index $j > 0$ represents the effect condition at a higher intensity. For instance, if $e$ represents the fever condition of a patient, it may have a domain $\{e^0, e^1, e^2\}$ which corresponds to

$$\{normal, low\ fever, high\ fever\}.$$

We will refer to $e^1, , ..., e^\eta$ as the *active* values of $e$, to $e^0$ as the *inactive* value of $e$, and to $e^\eta$ as the *most intensive* value of $e$. Notation $e < e^j$ is well defined, when $0 < j \leq \eta$, to denote $e \in \{e^0, e^1, ..., e^{j-1}\}$, and so is $e \geq e^j$, to denote $e \in \{e^j, e^{j+1}, ..., e^\eta\}$.

Let $c_i$ $(i = 1, 2, ...)$ be a multi-valued uncertain cause, whose finite domain is denoted $D_i = \{c_i^0, c_i^1, c_i^2, ...\}$. The value $c_i^0$ represents the absence of the condition signified by the variable $c_i$, and each value $c_i^j$ with a higher superscript index $j > 0$ represents the condition at a higher intensity. Variables such as $e$ and $c_i$ are often referred to as *graded* [7]. Although generally multi-valued variables are not necessarily graded, in this work, we assume they are. We will use *multi-valued* and *graded* interchangeably hereafter. We will refer to values of a graded cause variable as *active*, *inactive*, or *most intensive*, similarly to the way we refer to those of the effect.

We denote a set of multi-valued cause variables of effect $e$ (multi-valued) as $X = \{c_1, ..., c_n\}$. The set of *all causes* of $e$ is denoted by $C$. Set $C$ is assumed to include a leaky variable (if any) to capture causes not represented explicitly.

Causal events with multi-valued variables can be categorized from several perspectives. First, they can be categorized as *success* or *failure* events, depending on whether the effect is rendered active at a sufficiently high intensity. Second, they can be categorized as *single-causal* or *multi-causal*, depending on the number of active causes. Third, they can be categorized as *simple* or *congregate*, depending on the range of effect values involved. Below, we define causal events according to these categories precisely.

A *simple single-causal success* is an event that a cause $c_i$ with value $c_i^j$ $(j > 0)$ caused the effect $e$ to occur at a value $e^k$ $(k > 0)$, when every other cause $c_m$ of $e$ has the value $c_m^0$ (inactive). Condition $j > 0$ means that the cause $c_i$ must be active, and $k > 0$ says that the effect must be active. Denote this event by

$$e^k \leftarrow \{c_i^j\} \ \ \textit{or simply} \ \ \ e^k \leftarrow c_i^j.$$

The probability of the event is $P(e^k \leftarrow c_i^j)$, which we refer to as a *causal probability*, and it relates to conditional probability as

$$P(e^k \leftarrow c_i^j) = P(e^k | c_i^j, c_m^0 \ for\ each\ c_m \in C\ where\ m \neq i). \qquad (3)$$

A *congregate single-causal success* is an event that a cause $c_i$ with value $c_i^j$ $(j > 0)$ caused the effect $e$ to occur at a value $e^k$ $(k > 0)$ or higher, when every other cause is inactive. Denote this event by

$$e \geq e^k \leftarrow c_i^j.$$

The probability of the event is $P(e \geq e^k \leftarrow c_i^j)$, and it relates to conditional probability as

$$P(e \geq e^k \leftarrow c_i^j) = P(e \geq e^k | c_i^j, c_m^0 \textit{ for each } c_m \in C \textit{ where } m \neq i). \quad (4)$$

A *multi-causal success* involves a set $X = \{c_1, ..., c_n\}$ $(n > 1)$ of active causes of $e$, where each $c_i \in X$ has a value $c_i^j$ $(j > 0)$, when every other cause $c_m \in C \setminus X$ is inactive. In a *simple multi-causal success*, causes in $X$ collectively caused the effect $e$ to occur at $e^k$ $(k > 0)$. We denote the event by

$$e^k \leftarrow \{c_1^{j_1}, ..., c_n^{j_n}\} \quad \textit{or simply} \quad e^k \leftarrow c_1^{j_1}, ..., c_n^{j_n}$$

or by the (somewhat abused) vector notion

$$e^k \leftarrow \underline{x}^+,$$

where superscript + signifies that, for each $c_i \in X$, its value $c_i^{j_i} > c_i^0$. The corresponding causal probability relates to conditional probability as

$$P(e^k \leftarrow c_1^{j_1}, ..., c_n^{j_n}) = P(e^k | c_1^{j_1}, ..., c_n^{j_n}, c_m^0 \textit{ for each } c_m \in C \setminus X). \quad (5)$$

In a *congregate multi-causal success*, causes in $X$ collectively caused the effect $e$ to occur at $e^k$ $(k > 0)$ or higher. We denote the event by

$$e \geq e^k \leftarrow c_1^{j_1}, ..., c_n^{j_n} \quad \textit{or} \quad e \geq e^k \leftarrow \underline{x}^+.$$

The corresponding causal probability relates to conditional probability as

$$P(e \geq e^k \leftarrow c_1^{j_1}, ..., c_n^{j_n}) = P(e \geq e^k | c_1^{j_1}, ..., c_n^{j_n}, c_m^0 \textit{ for each } c_m \in C \setminus X). \quad (6)$$

A *congregate single-causal failure* refers to an event where $e < e^k$ $(k > 0)$ when a cause $c_i$ has a value $c_i^j$ $(j > 0)$ and every other cause $c_m$ is inactive. It is a failure in the sense that $c_i$ fails to produce the effect with an intensity $e^k$ or higher. We denote the failure event by

$$e < e^k \leftarrow c_i^j.$$

In the event of a *congregate multi-causal failure*, a set $X = \{c_1, ..., c_n\}$ $(n > 1)$ of causes are active while the effect $e < e^k$ $(k > 0)$. That is, $e < e^k$, each $c_i \in X$ has a value $c_i^j$ $(j > 0)$, and each $c_m \in C \setminus X$ has the value $c_m^0$. We denote the failure event by

$$e < e^k \leftarrow c_1^{j_1}, ..., c_n^{j_n} \quad \textit{or} \quad e < e^k \leftarrow \underline{x}^+.$$

Although *simple failure* events can be similarly defined, they are not referred to in this work. On the other hand, congregate causal events play an important role in generalizing NIN-AND tree causal models to multi-valued variables as will be seen in the subsequent sections.

The negation of congregate success $e \geq e^k \leftarrow c_1^{j_1}, ..., c_n^{j_n}$ is the congregate failure $e < e^k \leftarrow c_1^{j_1}, ..., c_n^{j_n}$ and vice versa. An equation that converts the two corresponding causal probabilities is given below in Proposition 1. Relation between causal probabilities of congregate failures and corresponding conditional probabilities can then be established through Eqns. (4) and (6).

When a cause takes a more intensive value, it may increase the probability of a causal success. For example, it may be the case that

$$P(e^k \leftarrow c_i^m) > P(e^k \leftarrow c_i^j), \quad whenever \quad m > j > 0.$$

Our interpretation of indexes of cause values allow such semantics. However, we make no such assumption in this work. That is, our results are applicable whether or not such semantics is adopted.

Note that our terminology on multi-valued causal events differs from those based on inhibitors or intermediate variables (ours is arguably simpler), e.g., [1, 2, 11], and is more coherent with those in [4, 5], although the latter deal with only binary variables. An idea similar to congregate causal events has been used in extending noisy-OR [12].

Eqns. (3) through (6) allow conversions between causal probabilities and conditional probabilities. The following proposition collects five additional equations for conversions between different causal probabilities. The nine equations together allow common conversions of probabilities over arbitrary causal events. We will refer to them as *conversion equations*.

**Proposition 1.** *Let $e$ be an effect, $X$ be a set of active causes of $e$, and $X$ be instantiated to $\underline{x}^+$. Then the following hold, where $k > 0$.*

$$P(e \geq e^k \leftarrow \underline{x}^+) = 1 - P(e < e^k \leftarrow \underline{x}^+). \tag{7}$$

$$P(e^0 \leftarrow \underline{x}^+) = 1 - P(e \geq e^1 \leftarrow \underline{x}^+). \tag{8}$$

$$P(e^\eta \leftarrow \underline{x}^+) = P(e \geq e^\eta \leftarrow \underline{x}^+). \tag{9}$$

$$For \quad k < \eta, \quad P(e^k \leftarrow \underline{x}^+) = P(e \geq e^k \leftarrow \underline{x}^+) - P(e \geq e^{k+1} \leftarrow \underline{x}^+). \tag{10}$$

$$P(e \geq e^k \leftarrow \underline{x}^+) = \sum_{j=k}^{\eta} P(e^j \leftarrow \underline{x}^+). \tag{11}$$

Eqn. (7) deals with negation of a congregate causal event. The next three obtain simple causal probabilities from congregate causal probabilities. Eqn. (8) concerns the inactive value of the effect, Eqn. (9) involves the most intensive value, and Eqn. (10) deals with the other active values. Eqn. (11) obtains a congregate causal probability from simple causal probabilities.

Proof:

Eqn. (7) follows from that $e < e^k \leftarrow \underline{x}^+$ is the negation of $e \geq e^k \leftarrow \underline{x}^+$.

Eqn. (8) follows from that the negation of $e \geq e^1 \leftarrow \underline{x}^+$ is $e^0 \leftarrow \underline{x}^+$.

Eqn. (9) follows from equivalence between $e \geq e^\eta$ and $e = e^\eta$.

Eqns. (10) and (11) follow from the sum rule of probability for mutually exclusive events. $\square$

We illustrate the practical usage of graded multi-causal events with an example on home renovation.

**Example 3.** *Surface enhancers protects and rejuvenates the color and appearance for stone and tile surfaces in kitchens and bathrooms. Four types of enhancers can be ordered. Each type has a low grade product and a high grade product. When multiple types of enhancers are applied to the same surface, for some combinations, the degree of surface enhancement is improved beyond that achievable when only one type is applied, while for others, the degree of enhancement is reduced. Denote the degree of surface enhancement by e with domain*

$$D_e = \{e^0, e^1, e^2\} = \{\textit{no enhancing}, \textit{slightly enhancing}, \textit{strongly enhancing}\}.$$

*Denote the application of type i enhancer ($i = 1, 2, 3, 4$) by $h_i$ with domain*

$$D_i = \{h_i^0, h_i^1, h_i^2\} = \{\textit{not applied}, \textit{apply low grade}, \textit{apply high grade}\}.$$

*The causal probability $P(e^2 \leftarrow h_1^2, h_3^1)$ would tell a contractor the chance to obtain strong surface enhancement when the high grade product of type 1 enhancer and the low grade product of type 3 enhancer are both applied to the surface to be renovated.*

Since the cause and effect variables in the above example are multi-valued, binary NIN-AND tree models are not applicable for assessment of $P(e^2 \leftarrow h_1^2, h_3^1)$ and other multi-causal probabilities. Graded multi-causal events introduced in this section and generalized NIN-AND tree models to be presented below will enable such assessment.

## 4. Generalized NIN-AND Gates

In this section, we generalize NIN-AND gates to graded variables. To generalize direct NIN-AND gates, we first extend the concepts on success conjunction and independence based on congregate causal successes.

**Definition 2.** *Disjoint sets of causes $W_1, ..., W_m$ of effect $e$ satisfy* **graded success conjunction** *iff*

$$e \geq e^k \leftarrow \underline{w}_1^+, ..., \underline{w}_m^+ = (e \geq e^k \leftarrow \underline{w}_1^+) \wedge ... \wedge (e \geq e^k \leftarrow \underline{w}_m^+),$$

*where $k > 0$.*

**Definition 3.** *Disjoint sets of causes $W_1, ..., W_m$ of effect $e$ satisfy* **graded success independence** *iff events*

$$e \geq e^k \leftarrow \underline{w}_1^+, \quad ..., \quad e \geq e^k \leftarrow \underline{w}_m^+$$

*are independent of each other, where $k > 0$. That is, the following equation holds,*

$$P(e \geq e^k \leftarrow \underline{w}_1^+, ..., \underline{w}_m^+) = P(e \geq e^k \leftarrow \underline{w}_1^+) \times ... \times P(e \geq e^k \leftarrow \underline{w}_m^+). \quad (12)$$

When the interaction of causes satisfies graded success conjunction and graded success independence, it can be depicted by a graphical model as shown in Fig. 4, where the set of active causes is $X = \{c_1, ..., c_n\}$. The success conjunc-

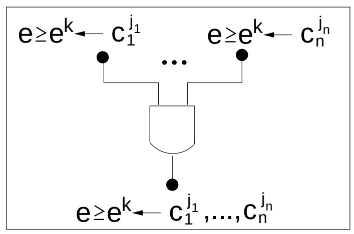


Figure 4: A generalized direct NIN-AND gate.

tion is represented by the AND gate. The success independence is signified by the disconnection of input events other than through the gate. Since the causes are uncertain causes, the AND gate is noisy. Common noisy-AND gates, e.g., [3], are *impeding* in that the probability of a causal success event is zero unless the set of active causes is equal to $C$. The probability of the output event of the gate in Fig. 4 is determined by Eqn. (12) from probabilities of the input events, whether $X = C$ or not. Hence, the gate is *non-impeding*. To distinguish it from the binary case (see Section 2) as well as the case introduced below, we term the gate in Fig. 4 as a *generalized direct non-impeding noisy-AND gate* or a generalized direct NIN-AND gate.

Next, we extend failure conjunction and independence, and generalize dual NIN-AND gates, based on congregate causal failures.

**Definition 4.** *Disjoint sets of causes $W_1, ..., W_m$ of effect $e$ satisfy* **graded failure conjunction** *iff*

$$e < e^k \leftarrow \underline{w}_1^+, ..., \underline{w}_m^+ = (e < e^k \leftarrow \underline{w}_1^+) \wedge ... \wedge (e < e^k \leftarrow \underline{w}_m^+),$$

*where $k > 0$.*

**Definition 5.** *Disjoint sets of causes $W_1, ..., W_m$ of effect $e$ satisfy* **graded failure independence** *iff failure events*

$$e < e^k \leftarrow \underline{w}_1^+, \quad ..., \quad e < e^k \leftarrow \underline{w}_m^+$$

*are independent of each other, where $k > 0$. That is, the following equation holds,*

$$P(e < e^k \leftarrow \underline{w}_1^+, ..., \underline{w}_m^+) = P(e < e^k \leftarrow \underline{w}_1^+) \times ... \times P(e < e^k \leftarrow \underline{w}_m^+). \quad (13)$$

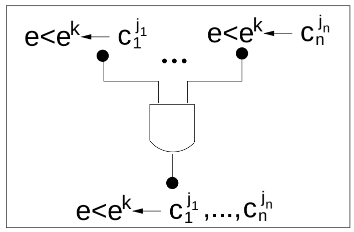


Figure 5: A generalized dual NIN-AND gate.

When the interaction of causes satisfies graded failure conjunction and graded failure independence, it can be depicted by a graphical model as shown in Fig. 5. The failure conjunction is represented by the AND gate, and the failure independence is signified by the disconnection of input events other than through the gate. The probability of the output event of the gate is determined by Eqn. (13) from probabilities of the input events. The gate in Fig. 5 differs from that in Fig. 4 in that all input and output events are causal failure events. Hence, we refer to it as a *generalized dual NIN-AND gate*.

Defs. 2 through 5 are applicable to sets of causes. Figs. 4 and 5 are special cases where these sets are singletons, reflected by the single-causal input events. In the general case, sets of active causes are involved, which will be reflected by multi-causal input events of the gate. A more general example appears in Fig. 6 below.

## 5. Reinforcing and Undermining Properties

This section analyzes the reinforcing and undermining behaviours of generalized NIN-AND gates, which generalize those of binary NIN-AND gates. First, we define reinforcing and undermining in the context of multi-valued effect.

**Definition 6.** *Let $e^k$ ($k > 0$) be an active value of effect $e$. Let $R = \{W_1, W_2, ...\}$ be a partition of a set $X$ of causes of $e$, $R' \subset R$ be any proper subset of $R$, and $Y = \cup_{W_i \in R'} W_i$. Sets of causes in $R$* **reinforce** *each other* **relative to** *$e^k$, iff*

$$\forall R' \; P(e \geq e^k \leftarrow \underline{y}^+) \leq P(e \geq e^k \leftarrow \underline{x}^+).$$

*Sets of causes in $R$* **undermine** *each other* **relative to** *$e^k$, iff*

$$\forall R' \; P(e \geq e^k \leftarrow \underline{y}^+) > P(e \geq e^k \leftarrow \underline{x}^+).$$

Note that Def. 6 is defined based on congregate causal probabilities rather than simple causal probabilities as Def. 1. The following proposition shows that a generalized direct NIN-AND gate models undermining.

**Proposition 2.** *Let $e^k$ $(k > 0)$ be an active value of effect $e$, $W_1, ..., W_m$ be disjoint sets of active causes of $e$ with their union $X = \cup_{i=1}^m W_i$,*

$$e \geq e^k \leftarrow \underline{w}_1^+, \quad ..., \quad e \geq e^k \leftarrow \underline{w}_m^+$$

*be input events of a generalized direct NIN-AND gate $g$, each associated with probability $P(e \geq e^k \leftarrow \underline{w}_i^+)$, and $P(e \geq e^k \leftarrow \underline{w}_1^+, ..., \underline{w}_m^+) = P(e \geq e^k \leftarrow \underline{x}^+)$ be probability associated with the output event of $g$.*

*Let $R'$ be a proper subset of $\{W_1, ..., W_m\}$ with its associated union $Y = \cup_{W_i \in R'} W_i$, $g'$ be another generalized direct NIN-AND gate with input events corresponding to elements of $R'$, and output event probability $P(e \geq e^k \leftarrow \underline{y}^+)$.*

*Then, the following hold:*

1. *For $i = 1, ..., m$, $P(e \geq e^k \leftarrow \underline{w}_1^+, ..., \underline{w}_m^+) < P(e \geq e^k \leftarrow \underline{w}_i^+)$.*
2. *$P(e \geq e^k \leftarrow \underline{x}^+) < P(e \geq e^k \leftarrow \underline{y}^+)$.*

Proof:

Both assertions follow from Eqn. (12). $\square$

**Example 4.** *Consider an example where $C = \{c_1, c_2\}$, $|D_1| = 2$, $|D_2| = |D_e| = 3$, and we are given the simple single-causal probabilities*

$$P(e^1 \leftarrow c_1^1) = 0.3, P(e^2 \leftarrow c_1^1) = 0.45,$$

$$P(e^1 \leftarrow c_2^1) = 0.35, P(e^2 \leftarrow c_2^1) = 0.22,$$

$$P(e^1 \leftarrow c_2^2) = 0.4, P(e^2 \leftarrow c_2^2) = 0.52.$$

*Let the input events of a generalized direct NIN-AND gate be*

$$e \geq e^2 \leftarrow c_1^1 \;\; \text{and} \;\; e \geq e^2 \leftarrow c_2^2,$$

*whose associated probabilities are 0.45 and 0.52, obtained from the above by conversion equations. The output event of the gate is $e \geq e^2 \leftarrow c_1^1, c_2^2$, and its associated probability is derived by Eqn. (12) as*

$$P(e \geq e^2 \leftarrow c_1^1, c_2^2) = 0.234.$$

*Let the input events of another generalized direct NIN-AND gate be*

$$e \geq e^1 \leftarrow c_1^1 \;\; \text{and} \;\; e \geq e^1 \leftarrow c_2^2,$$

*whose associated probabilities are 0.75 and 0.92. The output event of the gate is $e \geq e^1 \leftarrow c_1^1, c_2^2$, and its associated probability is derived as*

$$P(e \geq e^1 \leftarrow c_1^1, c_2^2) = 0.69.$$

*Note that the multi-causal probability of the output event obtained from each gate is less than the single-causal probabilities of input events: undermining interaction.*

Undermining is defined based on congregate causal probabilities (Def. 6). Would it manifest if examined based on simple causal probabilities? To answer this question, we apply conversion equations to

$$P(e \geq e^1|c_1^1, c_2^2) = 0.69, P(e \geq e^2|c_1^1, c_2^2) = 0.234,$$

and obtain conditional probabilities with the same active values for causes:

$$P(e^0|c_1^1, c_2^2) = 0.31, P(e^1|c_1^1, c_2^2) = 0.456, P(e^2|c_1^1, c_2^2) = 0.234.$$

They can be equivalently expressed as simple causal probabilities, e.g., $P(e^1 \leftarrow c_1^1, c_2^2) = 0.456$. It can be seen that undermining does not manifest if examined based on simple causal probabilities, e.g.,

$$P(e^1 \leftarrow c_1^1, c_2^2) = 0.456 > 0.3 = P(e^1 \leftarrow c_1^1),$$

and

$$P(e^1 \leftarrow c_1^1, c_2^2) = 0.456 > 0.4 = P(e^1 \leftarrow c_2^2).$$

The next proposition shows that a generalized dual NIN-AND gate models reinforcement.

**Proposition 3.** *Let $e^k$ $(k > 0)$ be an active value of effect $e$, $W_1, ..., W_m$ be disjoint sets of active causes of $e$ with their union $X = \cup_{i=1}^m W_i$,*

$$e < e^k \leftarrow \underline{w}_1^+, \quad ..., \quad e < e^k \leftarrow \underline{w}_m^+$$

*be input events of a generalized dual NIN-AND gate $g$, each associated with probability $P(e < e^k \leftarrow \underline{w}_i^+)$, and $P(e < e^k \leftarrow \underline{w}_1^+, ..., \underline{w}_m^+) = P(e < e^k \leftarrow \underline{x}^+)$ be probability associated with the output event of $g$.*

*Let $R'$ be a proper subset of $\{W_1, ..., W_m\}$ with its associated union $Y = \cup_{W_i \in R'} W_i$, $g'$ be another generalized dual NIN-AND gate with input events corresponding to elements of $R'$, and output event probability $P(e < e^k \leftarrow \underline{y}^+)$.*

*Then, the following hold:*

1. *For $i = 1, ..., m$, $P(e \geq e^k \leftarrow \underline{w}_1^+, ..., \underline{w}_m^+) > P(e \geq e^k \leftarrow \underline{w}_i^+)$.*
2. *$P(e \geq e^k \leftarrow \underline{x}^+) > P(e \geq e^k \leftarrow \underline{y}^+)$.*

Proof:

For the first statement, from Eqn. (13) for a generalized dual NIN-AND gate, we have, for $i = 1, ..., m$,

$$P(e < e^k \leftarrow \underline{w}_1^+, ..., \underline{w}_m^+) < P(e < e^k \leftarrow \underline{w}_i^+).$$

From conversion equations, the statement follows.

The second statement follows from Eqn. (13) and conversion equations with a similar argument. $\square$

**Example 5.** *Consider the effect and causes in the above example with the given single-causal probabilities, but apply to a dual gate. Let the input events of a generalized dual NIN-AND gate be* $e < e^2 \leftarrow c_1^1$ *and* $e < e^2 \leftarrow c_2^2$*, whose associated probabilities are 0.55 and 0.48, obtained by conversion equations. The output event of the gate is* $e < e^2 \leftarrow c_1^1, c_2^2$*, and its associated probability is derived by Eqn. (13) as* $P(e < e^2 \leftarrow c_1^1, c_2^2) = 0.264$*. Reinforcing interaction manifests as*

$$P(e \geq e^2 \leftarrow c_1^1, c_2^2) = 0.736 > \max(P(e \geq e^2 \leftarrow c_1^1) = 0.45, P(e \geq e^2 \leftarrow c_2^2) = 0.52).$$

*Using another dual gate with input events* $e < e^1 \leftarrow c_1^1$ *and* $e < e^1 \leftarrow c_2^2$*, whose associated probabilities are 0.25 and 0.08, we obtain the probability of its output event as* $P(e < e^1 \leftarrow c_1^1, c_2^2) = 0.02$.

Def. 6 defines reinforcement based on congregate causal probabilities. What would happen if we examine based on simple causal probabilities? From the above result $P(e < e^2 \leftarrow c_1^1, c_2^2) = 0.264$, $P(e < e^1 \leftarrow c_1^1, c_2^2) = 0.02$, and conversion equations, we obtain

$$P(e^0|c_1^1, c_2^2) = 0.02, P(e^1|c_1^1, c_2^2) = 0.244, P(e^2|c_1^1, c_2^2) = 0.736.$$

It can be seen that reinforcement does not manifest if examined based on simple causal probabilities, e.g.,

$$P(e^1 \leftarrow c_1^1, c_2^2) = P(e^1|c_1^1, c_2^2) = 0.244 < 0.3 = P(e^1 \leftarrow c_1^1).$$

In summary, a generalized direct NIN-AND gate expresses undermining and a generalized dual NIN-AND gate expresses reinforcement, judged based on congregate causal probabilities.

## 6. Generalized NIN-AND Trees

The following definition generalizes the binary NIN-AND tree models to multi-valued effects and causes. To facilitate comprehension, an intuitive explanation is given after each formal item.

**Definition 7.** *A* **generalized NIN-AND tree** *is a directed tree for a multi-valued effect* $e$ *and a set* $X = \{c_1, ..., c_n\}$ *of multi-valued causes, relative to a* **boundary** *value* $e^k$ $(k > 0)$ *of* $e$ *and an instantiation* $\underline{x}^+ = \{c_1^{j_1}, ..., c_n^{j_n}\}$ *of* $X$*, where* $j_i > 0$ $(i = 1, ..., n)$.

1. *There are two types of nodes. An* **event** *node (a black circle) has an in-degree* $\leq 1$ *and an out-degree* $\leq 1$*. A* **gate** *node (a generalized NIN-AND gate) has an in-degree* $\geq 2$ *and an out-degree* $1$.
   An event node corresponds to a causal event, and a gate node specifies the nature of causal interaction among its input events.

2. *There are two types of links, each connecting an event and a gate along the input-to-output direction of gates. A* **forward** *link (a line) is implicitly directed. A* **negation** *link (with a white circle at one end) is explicitly directed.*
   A forward link feeds a causal event directly into a gate, and a negation link negates a causal event before feeding it into a gate.
3. *Each terminal node is an event labelled by a graded causal event $e \geq e^k \leftarrow \underline{y}^+$ or $e < e^k \leftarrow \underline{y}^+$. There is a single* **leaf** *(no child) where $\underline{y}^+ = \underline{x}^+$, and the gate it connects to is the* **leaf gate***. For each* **root** *(no parent; indexed by $i$), $\underline{y}_i^+ \subset \underline{x}^+$, $\underline{y}_j^+ \cap \underline{y}_k^+ = \emptyset$ for $j \neq k$, and $\bigcup_i \underline{y}_i^+ = \underline{x}^+$.*
   Root nodes represent input causal events, and the leaf node represents the causal event due to interaction of all causes in the root nodes. Each cause appears in exactly one root node.
4. *Inputs to a gate $g$ are in one of two cases:*
   (a) *Each is either connected by a forward link to a node labelled $e \geq e^k \leftarrow \underline{y}^+$, or by a negation link to a node labelled $e < e^k \leftarrow \underline{y}^+$. The output of $g$ is connected by a forward link to a node labelled $e \geq e^k \leftarrow \cup_i \underline{y}_i^+$.*
   This involves a direct gate, whose input events are all causal success events, and so is its output event.
   (b) *Each is either connected by a forward link to a node labelled $e < e^k \leftarrow \underline{y}^+$, or by a negation link to a node labelled $e \geq e^k \leftarrow \underline{y}^+$. The output of $g$ is connected by a forward link to a node labelled $e < e^k \leftarrow \cup_i \underline{y}_i^+$.*
   This involves a dual gate, whose input events are all causal failure events, and so is its output event.
5. *Whenever two gate nodes are connected through an event node, their types (direct or dual) differ.*

Note that condition 5 ensures that each generalized NIN-AND tree encodes a unique causal interaction structure (see [13, 14] for details). Note also that the event associated with a root node may be single-causal or multi-causal, although this work focuses on root events that are single-causal only.

**Example 6.** *Fig. 6 is an example of a generalized NIN-AND tree for $C = \{c_1, c_2, c_3\}$ where $|D_e| = |D_1| = |D_2| = |D_3| = 3$.*

Next, we consider numerical parameters concerning root causal events in a generalized NIN-AND tree. It is possible that the probability associated with each root event (a congregate single-causal success or failure) is directly specified. Alternatively, only the probability of each simple single-causal success is directly specified.

**Example 7.** *With the example in Fig. 6, the following causal probabilities may be specified.*

$$P(e^1 \leftarrow c_1^1) = 0.20, P(e^2 \leftarrow c_1^1) = 0.40; \quad P(e^1 \leftarrow c_1^2) = 0.30, P(e^2 \leftarrow c_1^2) = 0.50;$$

$$P(e^1 \leftarrow c_2^1) = 0.25, P(e^2 \leftarrow c_2^1) = 0.35; \quad P(e^1 \leftarrow c_2^2) = 0.44, P(e^2 \leftarrow c_2^2) = 0.33;$$

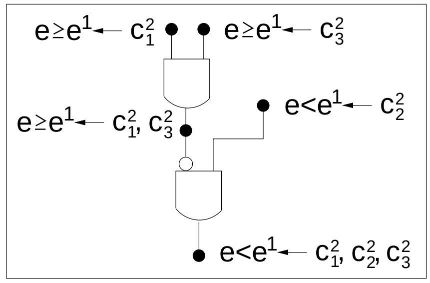


Figure 6: A generalized NIN-AND tree.

$$P(e^1 \leftarrow c_3^1) = 0.32, P(e^2 \leftarrow c_3^1) = 0.64; \quad P(e^1 \leftarrow c_3^2) = 0.16, P(e^2 \leftarrow c_3^2) = 0.60.$$

*In that case, the congregate causal probability associated with each root event must be obtained through conversion equations. For instance,* $P(e \geq e^1 \leftarrow c_1^2)$ *for the top left root node can be obtained as*

$$P(e^1 \leftarrow c_1^2) + P(e^2 \leftarrow c_1^2) = 0.80,$$

*and* $P(e < e^1 \leftarrow c_2^2)$ *for the middle right root node can be obtained as*

$$1 - P(e^1 \leftarrow c_2^2) - P(e^2 \leftarrow c_2^2) = 0.23.$$

Next, we consider evaluation of probability of the leaf event in a generalized NIN-AND tree. This can be performed using Algorithm 1 *GetCausalEventProb*. From the above example, we obtain

$$P(e < e^1 \leftarrow c_1^2, c_2^2, c_3^2) = 0.09016.$$

Finally, we consider evaluation of CPT for effect $e$ and its set $C$ of all causes. Before doing so, we introduce the concept of NIN-AND tree consistency.

**Definition 8.** *Let $T$ be a generalized NIN-AND tree over $e$ and $X$, and $T'$ be another generalized NIN-AND tree over $e$ and $X' \subseteq X$. $T$ and $T'$ are consistent iff one of the following holds.*

1. *$X' = X$: $T$ and $T'$ are isomorphic relative to variables. That is, they are structurally isomorphic with corresponding event node labels differing only in variable values.*
2. *$X' \subset X$: A generalized NIN-AND tree $T''$ over $e$ and $X'$ can be obtained by the following operation from $T$ such that $T''$ and $T'$ are isomorphic relative to variables.*
   *The operation consists of recursive removal of each root node $r$ (and its outgoing link) in $T$ if none of the causes for $r$ is in $X'$. If in-degree of a gate node $g$ is reduced to 0 as the result, remove $g$, its output event node, and links connected to them. If in-degree of $g$ is reduced to 1, replace its output event node with its input event node, and remove $g$ as well as links connected to $g$. Adjust link types and event labels accordingly.*

When a generalized NIN-AND tree over $X' \subseteq X$ is consistent with another over $X$, they express the same causal interaction among elements of $X'$. Fig. 7 (a) shows a generalized NIN-AND tree that is consistent with that in Fig. 6, both over $C$. Note that the boundary value of $e$ is $e^1$ in Fig. 6, but is $e^2$ in (a). Also, the value of $c_2$ is $c_2^2$ in Fig. 6, but is $c_2^1$ in (a).

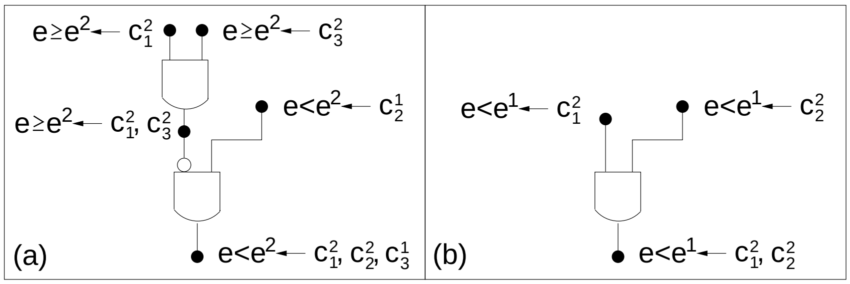


Figure 7: Generalized NIN-AND trees that are consistent with that of Fig. 6.

Fig. 7 (b) shows another generalized NIN-AND tree, over $X' = \{c_1, c_2\}$, that is consistent with that in Fig. 6. The top-right root node of Fig. 6 involves cause $c_3 \notin X'$. Its removal in turn causes the removal of the top gate node. The remaining input node of the gate replaces its output node, whose label as well as the type of its connecting link are adjusted, as shown in (b).

To evaluate CPT $P(e|C)$, a set of generalized NIN-AND trees is required, each corresponding to a unique combination of active boundary value $e^k$, a subset (not necessarily proper) $X$ of $C$, where $|X| \geq 2$, and an active instantiation of $X$. We assume that all of them are consistent, which is the typical case. Hence, it is sufficient to acquire a single generalized NIN-AND tree $T$ over $e$ and $C$, since all others can be derived from $T$. We also assume that for each combination of an active value of $e$ and an active value of a cause in $C$, the corresponding simple single-causal probability is acquired.

For each combination of active effect value $e^k$, cause subset $X = \{c_1, ..., c_n\} \subseteq C$, where $n > 1$, and active instantiation $\underline{x}^+ = (c_1^{j_1}, ..., c_n^{j_n})$, obtain its consistent generalized NIN-AND tree $T'$ from $T$. Derive causal probabilities $P(e \geq e^k \leftarrow c_i^j)$ or $P(e < e^k \leftarrow c_i^j)$ of its root events from the simple single-causal probability through conversion equations. Compute the leaf event probability $P(e \geq e^k \leftarrow \underline{x}^+)$ or $P(e < e^k \leftarrow \underline{x}^+)$ by algorithm *GetCausalEventProb(T')*.

After the above has been done for $k = 1, ..., \eta$, from

$$P(e \geq e^1 \leftarrow \underline{x}^+), ..., P(e \geq e^\eta \leftarrow \underline{x}^+)$$

and conversion equations, obtain simple causal probabilities,

$$P(e^1 \leftarrow \underline{x}^+), ..., P(e^\eta \leftarrow \underline{x}^+),$$

which define conditional probabilities by conversion equations,

$$P(e^1|\underline{x}^+, \underline{y}^0), ..., P(e^\eta|\underline{x}^+, \underline{y}^0),\ where\ Y = C \setminus X.$$

Algorithm 2 summarizes the above computation for CPT evaluation. Lines 1 through 9 obtain conditional probabilities where two or more active causes are involved. Lines 10 through 12 specify those of a single active cause. Lines 13 and 14 specify those where all causes are inactive.

**Algorithm 2.** *GetCptFromGenNinAndTree(T)*
*Input: A generalized NIN-AND tree $T$ over $e$ and $C$, and a set $SSCP$ of simple single-causal probabilities $\{P(e^k \leftarrow c_i^j)\}$, where $k > 0$, $c_i \in C$, and $j > 0$.*

1 *for each $X = \{c_1, ..., c_n\} \subseteq C$, where $n > 1$, do*
2    *for each $(c_1^{j_1}, ..., c_n^{j_n}) = \underline{x}^+$, where $j > 0$, do*
3       *for each $e^k$, where $k > 0$, do*
4          *get generalized NIN-AND tree $T'$ over $e$ and $X$ that is consistent with $T$;*
5          *get probabilities of root events of $T'$ from SSCP;*
6          *get probability $p$ of leaf event of $T'$ by GetCausalEventProb($T'$);*
7          *if $p$ is $P(e < e^k \leftarrow \underline{x}^+)$, obtain $P(e \geq e^k \leftarrow \underline{x}^+)$ from $p$;*
8       *get $P(e^k \leftarrow \underline{x}^+)$ $(k = 1, ..., \eta)$ from $P(e \geq e^k \leftarrow \underline{x}^+)$ $(k = 1, ..., \eta)$;*
9       *get $P(e^k|\underline{x}^+, \underline{y}^0) = P(e^k \leftarrow \underline{x}^+)$ $(k = 1, ..., \eta)$, where $Y = C \setminus X$;*

10 *for each $X = \{c_i\} \subseteq C$, do*
11    *for each $c_i^j$, where $j > 0$, do*
12       *$P(e^k|c_i^j, \underline{y}^0) = P(e^k \leftarrow c_i^j)$ $(k = 1, ..., \eta)$, where $Y = C \setminus X$;*

13 *$P(e^k|\underline{c}^0) = 0$ $(k = 1, ..., \eta)$, where $\underline{c}^0$ is the inactive instantiation of $C$;*
14 *return $P(e|C)$ where $e \neq e^0$;*

**Example 8.** *For the generalized NIN-AND tree in Fig. 6 and the single-causal probabilities specified above, the CPT evaluated by GetCptFromGenNinAndTree is shown in Table 2.*

## 7. Properties of Generalized NIN-AND Trees

In the following two theorems, we show that generalized NIN-AND trees model both reinforcement, undermining, and their multi-level mixtures correctly. Theorem 1 below establishes this relative to a single generalized NIN-AND tree.

**Theorem 1.** *Let $T$ be a generalized NIN-AND tree with boundary value $e^k$, where the probability for each root node event is specified in the range $(0, 1)$. Let $P(v)$ be returned by GetCausalEventProb.*

*Then $P(v)$ combines the given probabilities according to reinforcement and undermining expressed by the topology of $T$, with each generalized direct NIN-AND gate corresponding to undermining and each generalized dual NIN-AND gate corresponding to reinforcement, relative to $e^k$.*

Proof:

*GetCausalEventProb* first evaluates the output event for each gate node whose inputs are root events. If the root events are graded causal successes,

Table 2: CPT for generalized NIN-AND tree in Fig. 6

| | | |
|---|---|---|
| $P(e^0\|c_1^0,c_2^0,c_3^0)=1.0$ | $P(e^1\|c_1^0,c_2^0,c_3^0)=0.0$ | $P(e^2\|c_1^0,c_2^0,c_3^0)=0.0$ |
| $P(e^0\|c_1^0,c_2^0,c_3^1)=0.04000$ | $P(e^1\|c_1^0,c_2^0,c_3^1)=0.32$ | $P(e^2\|c_1^0,c_2^0,c_3^1)=0.64$ |
| $P(e^0\|c_1^0,c_2^0,c_3^2)=0.24000$ | $P(e^1\|c_1^0,c_2^0,c_3^2)=0.16$ | $P(e^2\|c_1^0,c_2^0,c_3^2)=0.6$ |
| $P(e^0\|c_1^0,c_2^1,c_3^0)=0.4$ | $P(e^1\|c_1^0,c_2^1,c_3^0)=0.25$ | $P(e^2\|c_1^0,c_2^1,c_3^0)=0.35$ |
| $P(e^0\|c_1^0,c_2^1,c_3^1)=0.01600$ | $P(e^1\|c_1^0,c_2^1,c_3^1)=0.21799$ | $P(e^2\|c_1^0,c_2^1,c_3^1)=0.76600$ |
| $P(e^0\|c_1^0,c_2^1,c_3^2)=0.09600$ | $P(e^1\|c_1^0,c_2^1,c_3^2)=0.16399$ | $P(e^2\|c_1^0,c_2^1,c_3^2)=0.74$ |
| $P(e^0\|c_1^0,c_2^2,c_3^0)=0.22999$ | $P(e^1\|c_1^0,c_2^2,c_3^0)=0.44$ | $P(e^2\|c_1^0,c_2^2,c_3^0)=0.33$ |
| $P(e^0\|c_1^0,c_2^2,c_3^1)=0.00919$ | $P(e^1\|c_1^0,c_2^2,c_3^1)=0.232$ | $P(e^2\|c_1^0,c_2^2,c_3^1)=0.7588$ |
| $P(e^0\|c_1^0,c_2^2,c_3^2)=0.05519$ | $P(e^1\|c_1^0,c_2^2,c_3^2)=0.21280$ | $P(e^2\|c_1^0,c_2^2,c_3^2)=0.732$ |
| $P(e^0\|c_1^1,c_2^0,c_3^0)=0.4$ | $P(e^1\|c_1^1,c_2^0,c_3^0)=0.2$ | $P(e^2\|c_1^1,c_2^0,c_3^0)=0.4$ |
| $P(e^0\|c_1^1,c_2^0,c_3^1)=0.42399$ | $P(e^1\|c_1^1,c_2^0,c_3^1)=0.32$ | $P(e^2\|c_1^1,c_2^0,c_3^1)=0.25600$ |
| $P(e^0\|c_1^1,c_2^0,c_3^2)=0.544$ | $P(e^1\|c_1^1,c_2^0,c_3^2)=0.21599$ | $P(e^2\|c_1^1,c_2^0,c_3^2)=0.24000$ |
| $P(e^0\|c_1^1,c_2^1,c_3^0)=0.15999$ | $P(e^1\|c_1^1,c_2^1,c_3^0)=0.23000$ | $P(e^2\|c_1^1,c_2^1,c_3^0)=0.61$ |
| $P(e^0\|c_1^1,c_2^1,c_3^1)=0.16960$ | $P(e^1\|c_1^1,c_2^1,c_3^1)=0.31399$ | $P(e^2\|c_1^1,c_2^1,c_3^1)=0.51640$ |
| $P(e^0\|c_1^1,c_2^1,c_3^2)=0.21759$ | $P(e^1\|c_1^1,c_2^1,c_3^2)=0.27639$ | $P(e^2\|c_1^1,c_2^1,c_3^2)=0.50600$ |
| $P(e^0\|c_1^1,c_2^2,c_3^0)=0.09200$ | $P(e^1\|c_1^1,c_2^2,c_3^0)=0.30999$ | $P(e^2\|c_1^1,c_2^2,c_3^0)=0.59800$ |
| $P(e^0\|c_1^1,c_2^2,c_3^1)=0.09751$ | $P(e^1\|c_1^1,c_2^2,c_3^1)=0.40095$ | $P(e^2\|c_1^1,c_2^2,c_3^1)=0.50152$ |
| $P(e^0\|c_1^1,c_2^2,c_3^2)=0.12512$ | $P(e^1\|c_1^1,c_2^2,c_3^2)=0.38407$ | $P(e^2\|c_1^1,c_2^2,c_3^2)=0.49080$ |
| $P(e^0\|c_1^2,c_2^0,c_3^0)=0.19999$ | $P(e^1\|c_1^2,c_2^0,c_3^0)=0.3$ | $P(e^2\|c_1^2,c_2^0,c_3^0)=0.5$ |
| $P(e^0\|c_1^2,c_2^0,c_3^1)=0.232$ | $P(e^1\|c_1^2,c_2^0,c_3^1)=0.448$ | $P(e^2\|c_1^2,c_2^0,c_3^1)=0.32$ |
| $P(e^0\|c_1^2,c_2^0,c_3^2)=0.39200$ | $P(e^1\|c_1^2,c_2^0,c_3^2)=0.30799$ | $P(e^2\|c_1^2,c_2^0,c_3^2)=0.3$ |
| $P(e^0\|c_1^2,c_2^1,c_3^0)=0.07999$ | $P(e^1\|c_1^2,c_2^1,c_3^0)=0.245$ | $P(e^2\|c_1^2,c_2^1,c_3^0)=0.675$ |
| $P(e^0\|c_1^2,c_2^1,c_3^1)=0.09280$ | $P(e^1\|c_1^2,c_2^1,c_3^1)=0.3492$ | $P(e^2\|c_1^2,c_2^1,c_3^1)=0.55799$ |
| $P(e^0\|c_1^2,c_2^1,c_3^2)=0.15680$ | $P(e^1\|c_1^2,c_2^1,c_3^2)=0.29819$ | $P(e^2\|c_1^2,c_2^1,c_3^2)=0.545$ |
| $P(e^0\|c_1^2,c_2^2,c_3^0)=0.04600$ | $P(e^1\|c_1^2,c_2^2,c_3^0)=0.28899$ | $P(e^2\|c_1^2,c_2^2,c_3^0)=0.665$ |
| $P(e^0\|c_1^2,c_2^2,c_3^1)=0.05335$ | $P(e^1\|c_1^2,c_2^2,c_3^1)=0.40223$ | $P(e^2\|c_1^2,c_2^2,c_3^1)=0.54440$ |
| $P(e^0\|c_1^2,c_2^2,c_3^2)=0.09016$ | $P(e^1\|c_1^2,c_2^2,c_3^2)=0.37883$ | $P(e^2\|c_1^2,c_2^2,c_3^2)=0.531$ |

then the gate is a generalized direct NIN-AND gate. By Proposition 2, the probability of the output event reflects the result of undermining, relative to $e^k$. Otherwise, the root events are graded causal failures, and the gate is a generalized dual NIN-AND gate. By Proposition 3, the probability of the output event reflects the result of reinforcement, relative to $e^k$.

After the evaluation, root nodes, gate nodes connected to them, and links incident to both no longer participate in further evaluations and can be deleted. The remaining subtree is still a generalized NIN-AND tree with the depth reduced by one. Note that the depth of the tree is the maximum number of gate nodes on a path starting at the leaf node. *GetCausalEventProb* repeats the above computation until the depth reduces to zero. The statement is true for the evaluation performed at each depth and hence the theorem holds. $\square$

Theorem 2 below establishes the soundness relative to a family of consistent, generalized NIN-AND trees that collectively defines the CPT over an effect and its causes.

**Theorem 2.** *Let $T$ be a generalized NIN-AND tree over effect $e$ and its set $C$ of all causes. Let $SSCP$ be a set of all simple single-causal probabilities, each over an active $e^k$ and an active cause of $C$. Let $P(e|C)$ be the CPT returned by GetCptFromGenNinAndTree.*

*Then for every active $e^k$ and every active instantiation $\underline{x}^+$ of $X \subseteq C$, $P(e|\underline{x}^+)$ models reinforcement and undermining among elements of $X$ that are expressed by the topology of $T$.*

Proof:

For each $\underline{x}^+$, the distribution $P(e|\underline{x}^+)$ is sufficient to determine $P(e \geq e^k \leftarrow \underline{x}^+)$ for $k = 1, ..., \eta$. For each $e^k$, $P(e \geq e^k \leftarrow \underline{x}^+)$ models reinforcement and undermining among elements of $X$ that are expressed by the topology of a generalized NIN-AND tree $T'$ according to Theorem 1. Since $T'$ is consistent with $T$ according to *GetCptFromGenNinAndTree*, $P(e \geq e^k \leftarrow \underline{x}^+)$ models reinforcement and undermining among elements of $X$ that are expressed by the topology of $T$. □

Next, we consider the efficiency issue for acquisition of a generalized NIN-AND tree model. Acquisition of $T$, a generalized NIN-AND tree over $e$ and $C$, is efficient as $T$ has $O(n)$ nodes and links, where $n = |C|$. The following theorem establishes that acquisition of required numerical parameters is also efficient.

**Theorem 3.** *Let $C = \{c_1, ..., c_n\}$ be the set of all causes of effect $e$ and the CPT $P(e|C)$ is to be evaluated by GetCptFromGenNinAndTree. Denote $|D_e|$ by $\eta+1$ and $|D_i|$ by $\beta_i+1$ $(i = 1, ..., n)$. Then the complexity of acquisition for numerical parameters required by GetCptFromGenNinAndTree is $O(\eta\ (\beta_1 + ... + \beta_n))$.*

Proof:

The task of acquisition for numerical parameters is to acquire the set SSCP. Each simple single-causal probability in SSCP involves a unique combination of an active $e^k$ and an active cause $c_i^j$. Hence, the statement follows. □

Assuming $\eta = \beta_i$ for $i = 1, ..., n$, the above complexity becomes $O(n\ \eta^2)$ and is hence linear in $n$.

The following proposition shows that binary NIN-AND tree models are special cases of generalized NIN-AND tree models. That is, a generalized NIN-AND tree model with all variables being binary degenerates to a binary NIN-AND tree model.

**Proposition 4.** *Let $T$ be a generalized NIN-AND tree over effect $e$ and its set $C$ of all causes, where all variables are binary, $SSCP$ be a set of all simple single-causal probabilities, each over an active $e^k$ and an active cause of $C$.*

*Then $T$ and $SSCP$ defines a binary NIN-AND tree model.*

Proof:

Denote $C = \{c_1, ..., c_n\}$. First, we consider the causal interaction structure $T$. Since all variables are binary, for each root node, the causal event is either $e \geq e^1 \leftarrow c_i^1$ or $e < e^1 \leftarrow c_i^1$. Event $e \geq e^1 \leftarrow c_i^1$ is equivalent to $e^+ \leftarrow c_i^+$, and $e < e^1 \leftarrow c_i^1$ is equivalent to $e^- \leftarrow c_i^+$, which are events associated with a binary NIN-AND tree. This equivalence also leads to the conclusion that generalized direct and dual NIN-AND gates degenerate to (binary) direct and dual NIN-AND gates.

When all variables are binary, $SSCP$ consists of

$$P(e^1 \leftarrow c_i^1) \ \ (n = 1, 2, ..., n).$$

They correspond exactly to the numerical parameters

$$P(e^+ \leftarrow c_i^+) \ \ (n = 1, 2, ..., n)$$

as required by a binary NIN-AND tree model over $e$ and $C$. Hence, the statement holds. $\square$

Given an effect $e$ and its set $C$ of all causes in a practical application domain, to construct a NIN-AND tree model, both the NIN-AND tree topology and causal probabilities of the root events must be obtained. The causal probabilities can be obtained using its relation with conditional probabilities. A direct approach to elicit the tree topology requires a domain expert to specify a partial order in which causes interact in terms of either reinforcement (a dual gate on the tree) or undermining (a direct gate). A number of alternative approaches have also been developed that either aid topology elicitation with an automated tool or indirectly elicit the topology [14, 15].

## 8. Conclusion

In this contribution, we generalize the binary NIN-AND tree causal models to multi-valued effects and causes. Generalized NIN-AND trees model explicitly reinforcement and undermining among causes, as well as their mixture at multiple levels, relative to each active value of the effect. Acquisition of a generalized NIN-AND tree model is shown to be efficient (Theorem 3). Hence, this contribution will allow CPTs of discrete Bayesian networks of arbitrary variables (binary or multi-valued) to be specified efficiently through the intuitive concepts of reinforcement and undermining.

The focus of algorithm *GetCptFromGenNinAndTree* presented is on how to evaluate CPT from a generalized NIN-AND tree model, rather than how to make the computation most efficient. Methods have been developed to improve efficiency in computing CPT from a binary NIN-AND tree model [8]. How to apply them for efficiency gain in *GetCptFromGenNinAndTree* requires future investigation.

In this work, we focus on generalized NIN-AND tree where all root nodes are single-causal. This is equivalent to an assumption that every pair of causes interact according to the success or failure conjunction and independence as specified by the tree structure. This assumption, however, can be relaxed when necessary. Future work is needed to formalize the relaxation.

**Acknowledgements**

The author thanks comments from anonymous reviewers. Financial support from NSERC, Canada is acknowledged.

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