C

Fourier Transform (FT) in the Field of DIP

FT in the Field of DIP

I. The 1-D Discrete FT (DFT): Introduction

I.1a. A Vector Space and Two Bases

$$\boldsymbol{W} \doteq \left\{ \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\}_{(\alpha,\beta)\in\mathbb{R}^2}$$

$$+ \left| \begin{array}{l} W^2 \to W \\ \left(\begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} \right) \mapsto \begin{bmatrix} \alpha + \alpha' \\ \beta + \beta' \end{bmatrix} \end{array} \right.$$

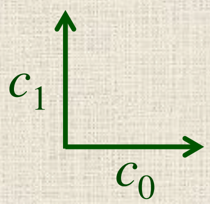
$$\bullet \left| \begin{array}{l} \mathbb{R} \times W \to W \\ \left(\lambda, \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right) \mapsto \begin{bmatrix} \lambda\alpha \\ \lambda\beta \end{bmatrix} \end{array} \right.$$

I.1b. A Vector Space and Two Bases

$(W,+,\bullet)$ is a real **vector space**.

$$\boldsymbol{c}_0 \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{c}_1 \doteq \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{array}{ll} c_0(0) = 1 & c_1(0) = 0 \\ c_0(1) = 0 & c_1(1) = 1 \end{array}$$

$\boldsymbol{c} \doteq (c_0, c_1)$ is the **canonical basis**.

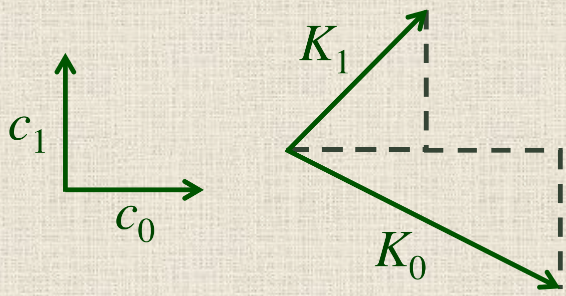


I.1c. A Vector Space and Two Bases

$(W,+,\cdot)$ is a real **vector space**.

$$K_0 \doteq \begin{bmatrix} 2 \\ -1 \end{bmatrix},\ K_1 \doteq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{matrix} K_0(0)=2 & K_1(0)=1 \\ K_0(1)=-1 & K_1(1)=1 \end{matrix}$$

$K \doteq (K_0, K_1)$ is a **basis**.

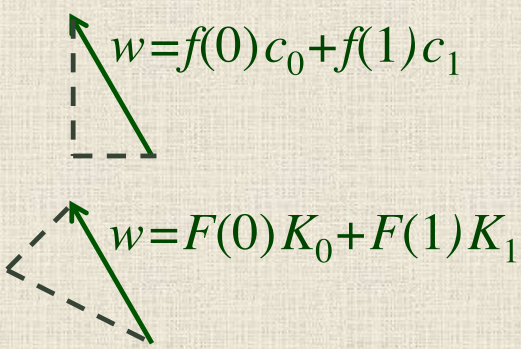


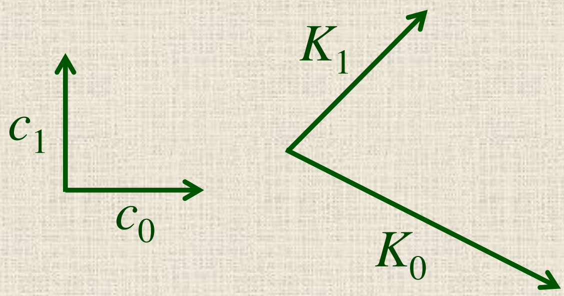
I.2a. Change of Basis

$w \doteq$ any vector

$f \doteq \begin{bmatrix} f(0) \\ f(1) \end{bmatrix} \doteq$ coordinates of w with respect to c

$F \doteq \begin{bmatrix} F(0) \\ F(1) \end{bmatrix} \doteq$ coordinates of w with respect to K





I.2b. Change of Basis

$\boldsymbol{w} \doteq$ any vector

$\boldsymbol{f} \doteq \begin{bmatrix} f(0) \\ f(1) \end{bmatrix} \doteq$ coordinates of w with respect to c

$w = f(0)c_0 + f(1)c_1$

$\boldsymbol{F} \doteq \begin{bmatrix} F(0) \\ F(1) \end{bmatrix} \doteq$ coordinates of w with respect to K

$w = F(0)K_0 + F(1)K_1$

Express $f(x)$ in terms of $F(u)$.
Express $F(u)$ in terms of $f(x)$.

I.2c. Change of Basis

$$\begin{bmatrix} f(0) \\ f(1) \end{bmatrix} = \begin{bmatrix} 2F(0) + F(1) \\ -F(0) + F(1) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} F(0) \\ F(1) \end{bmatrix}$$

$$\begin{bmatrix} F(0) \\ F(1) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} f(0) \\ f(1) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}$$

$$\begin{bmatrix} F(0) \\ F(1) \end{bmatrix} = \begin{bmatrix} f(0)/3 - f(1)/3 \\ f(0)/3 + 2f(1)/3 \end{bmatrix}$$

I.2d. Change of Basis

$$f = F(0)\begin{bmatrix} ? \end{bmatrix} + F(1)\begin{bmatrix} ? \end{bmatrix}$$

$$\begin{bmatrix} f(0) \\ f(1) \end{bmatrix} = \begin{bmatrix} 2F(0)+F(1) \\ -F(0)+F(1) \end{bmatrix}$$

$$F = f(0)\begin{bmatrix} ? \end{bmatrix} + f(1)\begin{bmatrix} ? \end{bmatrix}$$

$$\begin{bmatrix} F(0) \\ F(1) \end{bmatrix} = \begin{bmatrix} f(0)/3 - f(1)/3 \\ f(0)/3 + 2f(1)/3 \end{bmatrix}$$

I.2e. Change of Basis

$$f = F(0)\overset{K_0}{\begin{bmatrix} 2 \\ -1 \end{bmatrix}} + F(1)\overset{K_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$\begin{bmatrix} f(0) \\ f(1) \end{bmatrix} = \begin{bmatrix} 2F(0)+F(1) \\ -F(0)+F(1) \end{bmatrix}$$

$$F = f(0)\overset{\boldsymbol{k_0}}{\begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}} + f(1)\overset{\boldsymbol{k_1}}{\begin{bmatrix} -1/3 \\ 2/3 \end{bmatrix}}$$

$$\begin{bmatrix} F(0) \\ F(1) \end{bmatrix} = \begin{bmatrix} f(0)/3 - f(1)/3 \\ f(0)/3 + 2f(1)/3 \end{bmatrix}$$

I.2f. Change of Basis

$$f = F(0)\overset{K_0}{\begin{bmatrix} 2 \\ -1 \end{bmatrix}} + F(1)\overset{K_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$\begin{bmatrix} f(0) \\ f(1) \end{bmatrix} = \begin{bmatrix} F(0)K_0(0) + F(1)K_1(0) \\ F(0)K_0(1) + F(1)K_1(1) \end{bmatrix}$$

$$F = f(0)\overset{\mathbf{k_0}}{\begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}} + f(1)\overset{\mathbf{k_1}}{\begin{bmatrix} -1/3 \\ 2/3 \end{bmatrix}}$$

$$\begin{bmatrix} F(0) \\ F(1) \end{bmatrix} = \begin{bmatrix} f(0)k_0(0) + f(1)k_1(0) \\ f(0)k_0(1) + f(1)k_1(1) \end{bmatrix}$$

I.2g. Change of Basis

$$f = F(0)K_0 + F(1)K_1$$

$$\begin{bmatrix} f(0) \\ f(1) \end{bmatrix} = \begin{bmatrix} F(0)K_0(0) + F(1)K_1(0) \\ F(0)K_0(1) + F(1)K_1(1) \end{bmatrix}$$

$$F = f(0)k_0 + f(1)k_1$$

$$\begin{bmatrix} F(0) \\ F(1) \end{bmatrix} = \begin{bmatrix} f(0)k_0(0) + f(1)k_1(0) \\ f(0)k_0(1) + f(1)k_1(1) \end{bmatrix}$$

I.2h. Change of Basis

$$f = \sum_u F(u)\, K_u$$

$$\begin{bmatrix} f(0) \\ f(1) \end{bmatrix} = \begin{bmatrix} F(0)K_0(0) + F(1)K_1(0) \\ F(0)K_0(1) + F(1)K_1(1) \end{bmatrix}$$

$$F = \sum_x f(x)\, k_x$$

$$\begin{bmatrix} F(0) \\ F(1) \end{bmatrix} = \begin{bmatrix} f(0)k_0(0) + f(1)k_1(0) \\ f(0)k_0(1) + f(1)k_1(1) \end{bmatrix}$$

I.3a. DIP Terminology

f → **original image**

$$f = \sum_u F(u)\, \boxed{K_u}$$

K_u → they define the **inverse transformation kernel**

$$f(x) = \sum_u F(u)\, K_u(x)$$

F → **transformed image**

$$F = \sum_x f(x)\, \boxed{k_x}$$

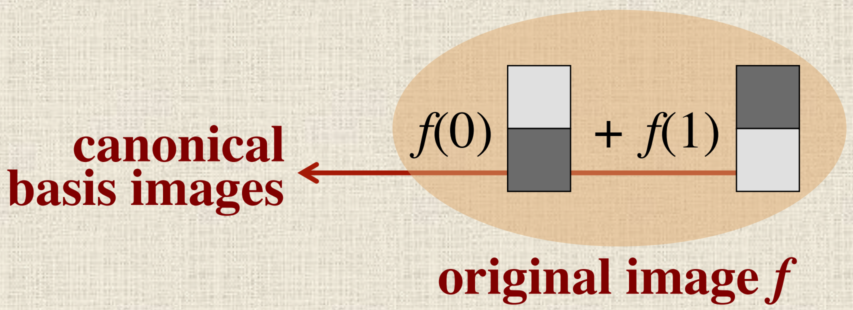
k_x → they define the **forward transformation kernel**

$$F(u) = \sum_x f(x)\, k_x(u)$$

I.3b. DIP Terminology

Canonical basis:

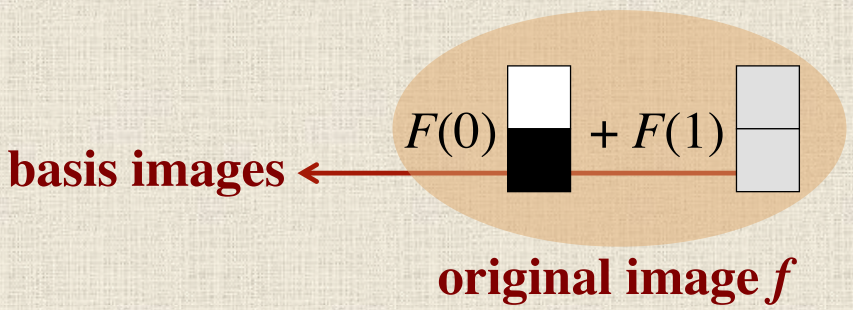
$$c_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ c_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow \textbf{canonical basis functions}$$



I.3c. DIP Terminology

Inverse transformation kernel:

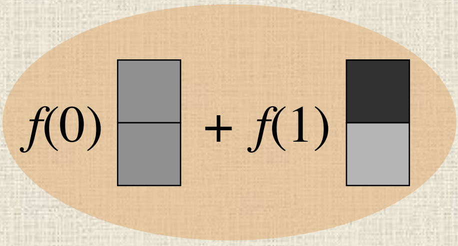
$$K_0 = \begin{bmatrix} 2 \\ -1 \end{bmatrix},\ K_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow \textbf{basis functions}$$



I.3d. DIP Terminology

Forward transformation kernel:

$$k_0 = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}, \; k_1 = \begin{bmatrix} -1/3 \\ 2/3 \end{bmatrix}$$

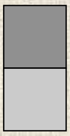
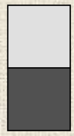


transformed image *F*

I.4a. Generalization

$f = \Sigma_u F(u)\, K_u$ $\qquad$ $F = \Sigma_x f(x)\, k_x$

I.4b. Generalization

$f = \Sigma_{u=0}^{1} F(u)\, K_u =$  $F = \Sigma_{x=0}^{1} f(x)\, k_x =$ 

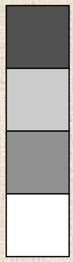
$(W,+,\cdot)$ is a real **vector space** (dimension 2).

$$\boldsymbol{W} \doteq \left\{ \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\}_{(\alpha,\beta)\in\mathbb{R}^2}$$

$+\ |\ W^2 \to W$

$\bullet\ |\ \mathbb{R} \times W \to W$

I.4c. Generalization

$f = \Sigma_{u=0}^{M-1} F(u)\, K_u =$  $F = \Sigma_{x=0}^{M-1} f(x)\, k_x =$ 

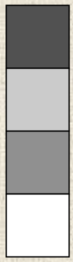
$(W,+,\cdot)$ is a real **vector space** (dimension M).

$$\boldsymbol{W} \doteq \left\{ \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \dots \\ \alpha_{M-1} \end{bmatrix} \right\}_{(\alpha_0,\alpha_1,\dots,\alpha_{M-1})\in\mathbb{R}^M}$$

$+\ |\ W^2 \to W$

$\bullet\ |\ \mathbb{R} \times W \to W$

I.4d. Generalization

$f = \Sigma_{u=0}^{M-1} F(u)\, K_u =$  $F = \Sigma_{x=0}^{M-1} f(x)\, k_x =$ 

$(W,+,\cdot)$ is a complex **vector space** (dimension M).

$$\boldsymbol{W} \doteq \left\{ \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \cdots \\ \alpha_{M-1} \end{bmatrix} \right\}_{(\alpha_0, \alpha_1, \ldots, \alpha_{M-1}) \in \mathbb{C}^M}$$

$$+ \,|\, W^2 \to W$$

$$\cdot \,|\, \mathbb{C} \times W \to W$$

I.4e. Generalization

$f = \Sigma_{u=0}^{M-1} F(u)\, K_u$ $F = \Sigma_{x=0}^{M-1} f(x)\, k_x$

Example: $K_u(x) \doteq \dfrac{1}{M} e^{2j\pi ux/M}$

↓

Fourier basis functions

II. The 1-D DFT: Definitions and Properties

II.1a. Definitions

$M \doteq$ positive integer

function $\doteq$ total function from $0..\, M$-1 into $\mathbb{C}$.

II.1b. Definitions

For any u and x in 0.. M-1:

$$K_u(x) \doteq \frac{1}{M} e^{2j\pi ux/M}$$

K is the **inverse Fourier transformation kernel**.

$$k_x(u) \doteq e^{-2j\pi ux/M}$$

k is the **forward Fourier transformation kernel**.

II.1c. Definitions

Consider a function F.
The function $f = \sum_{u=0}^{M-1} F(u) K_u$
is the **inverse Fourier transform of F**.
It is denoted by $\mathcal{F}^{-1}(F)$.

Consider a function f.
The function $F = \sum_{x=0}^{M-1} f(x) k_x$
is the **(forward) Fourier transform of f**.
It is denoted by $\mathcal{F}(f)$.

$\mathcal{F}$ is the **Fourier transform**.

II.2a. Justification

Consider two functions f and F.

$\mathcal{F}^{-1}(\mathcal{F}(f)) = f$

(i.e., if $F = \mathcal{F}(f)$ then $f = \mathcal{F}^{-1}(F)$)

$\mathcal{F}(\mathcal{F}^{-1}(F)) = F$

(i.e., if $f = \mathcal{F}^{-1}(F)$ then $F = \mathcal{F}(f)$)

II.2b. Justification

$$\mathcal{F}^{-1}(\mathcal{F}(f))(x)$$

$$= \frac{1}{M}\sum_{u=0}^{M-1}\left(\sum_{z=0}^{M-1} f(z)e^{-2j\pi uz/M}\right)e^{2j\pi ux/M}$$

$$= \frac{1}{M}\sum_{z=0}^{M-1} f(z)\left(\sum_{u=0}^{M-1} e^{2j\pi u(x-z)/M}\right)$$

$$W \doteq e^{2j\pi(x-z)/M}$$

If $z=x$ then $\sum_{u=0}^{M-1} W^u = M$

If $z \neq x$ then $\sum_{u=0}^{M-1} W^u = \frac{1-W^M}{1-W} = 0$

QED

II.3a. Basic Properties

Consider a function F: $\mathcal{F}^{-1}(F) = \frac{1}{M}\mathcal{F}(F^*)^*$

$$f \doteq \mathcal{F}^{-1}(F)$$

$$f(x) = \frac{1}{M}\sum_{u=0}^{M-1} F(u)e^{2j\pi ux/M}$$

$$f^*(x) = \frac{1}{M}\sum_{u=0}^{M-1} F^*(u)e^{-2j\pi ux/M}$$

$$f^*(x) = \frac{1}{M}\mathcal{F}(F^*)(x)$$

$$f = \frac{1}{M}\mathcal{F}(F^*)^*$$

QED

II.3b. Basic Properties

$\mathcal{F}$ is **linear**:

for any functions f_1 and f_2,
for any complex number c,
$\mathcal{F}(cf_1) = c\mathcal{F}(f_1)$ and $\mathcal{F}(f_1+f_2) = \mathcal{F}(f_1)+\mathcal{F}(f_2)$.

II.3c. Basic Properties

Let f be a function.

$$\hat{F} \left| \begin{array}{l} \mathbb{Z} \to \mathbb{C} \\ u \mapsto \sum_{x=0}^{M-1} f(x) e^{-2j\pi ux/M} \end{array} \right.$$

is **periodic**: $\forall u \in \mathbb{Z},\ \hat{F}(u+M) = \hat{F}(u)$

Moreover, **if f is a real function then** $\forall u \in \mathbb{Z},\ \hat{F}(-u) = \hat{F}^*(u)$

II.3d. Basic Properties

Let f be a function and let F be its FT.
Let u_0 be an integer.

The FT of $x \mapsto f(x) e^{2j\pi u_0 x/M}$ **is** $u \mapsto \hat{F}(u - u_0)$

If M is even, the FT of $x \mapsto (-1)^x f(x)$ **is** $u \mapsto \hat{F}(u - M/2)$

III. The 2-D DFT: Introduction

III.1a. A Vector Space and Two Bases

$$\mathbf{W} \doteq \left\{ \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \right\}_{(\alpha,\beta,\gamma,\delta)\in\mathbb{C}^4}$$

$$+ \left| \begin{array}{l} W^2 \to W \\ \left(\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}, \begin{bmatrix} \alpha' & \gamma' \\ \beta' & \delta' \end{bmatrix} \right) \mapsto \begin{bmatrix} \alpha+\alpha' & \gamma+\gamma' \\ \beta+\beta' & \delta+\delta' \end{bmatrix} \end{array} \right.$$

$$\bullet \left| \begin{array}{l} \mathbb{C}\times W \to W \\ \left(\lambda, \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \right) \mapsto \begin{bmatrix} \lambda\alpha & \lambda\gamma \\ \lambda\beta & \lambda\delta \end{bmatrix} \end{array} \right.$$

III.1b. A Vector Space and Two Bases

$(W,+,\cdot)$ is a complex **vector space**.

$$\boldsymbol{c}_{0,0} \doteq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \boldsymbol{c}_{0,1} \doteq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\boldsymbol{c}_{1,0} \doteq \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \boldsymbol{c}_{1,1} \doteq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\boldsymbol{c} \doteq (c_{0,0}, c_{0,1}, c_{1,0}, c_{1,1})$ is the **canonical basis**.

$$\begin{array}{llll} c_{0,0}(0,0)=1 & c_{0,1}(0,0)=0 & c_{1,0}(0,0)=0 & c_{1,1}(0,0)=0 \\ c_{0,0}(0,1)=0 & c_{0,1}(0,1)=1 & c_{1,0}(0,1)=0 & c_{1,1}(0,1)=0 \\ c_{0,0}(1,0)=0 & c_{0,1}(1,0)=0 & c_{1,0}(1,0)=1 & c_{1,1}(1,0)=0 \\ c_{0,0}(1,1)=0 & c_{0,1}(1,1)=0 & c_{1,0}(1,1)=0 & c_{1,1}(1,1)=1 \end{array}$$

III.1c. A Vector Space and Two Bases

$(W,+,\cdot)$ is a complex **vector space**.

$$\boldsymbol{K}_{0,0} \doteq \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \boldsymbol{K}_{0,1} \doteq \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\boldsymbol{K}_{1,0} \doteq \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \boldsymbol{K}_{1,1} \doteq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\boldsymbol{K} \doteq (K_{0,0}, K_{0,1}, K_{1,0}, K_{1,1})$ is a **basis**.

$$\begin{array}{llll} K_{0,0}(0,0)=1 & K_{0,1}(0,0)=1 & K_{1,0}(0,0)=1 & K_{1,1}(0,0)=1 \\ K_{0,0}(0,1)=1 & K_{0,1}(0,1)=1 & K_{1,0}(0,1)=0 & K_{1,1}(0,1)=0 \\ K_{0,0}(1,0)=1 & K_{0,1}(1,0)=0 & K_{1,0}(1,0)=1 & K_{1,1}(1,0)=0 \\ K_{0,0}(1,1)=1 & K_{0,1}(1,1)=0 & K_{1,0}(1,1)=0 & K_{1,1}(1,1)=1 \end{array}$$

III.2a. Change of Basis

$\boldsymbol{w} \doteq$ any vector

$$\boldsymbol{f} \doteq \begin{bmatrix} f(0,0) & f(0,1) \\ f(1,0) & f(1,1) \end{bmatrix} \doteq \text{coordinates of } w \text{ with respect to } c$$

$$\boldsymbol{F} \doteq \begin{bmatrix} F(0,0) & F(0,1) \\ F(1,0) & F(1,1) \end{bmatrix} \doteq \text{coordinates of } w \text{ with respect to } K$$

Express $f(x,y)$ in terms of $F(u,v)$.
Express $F(u,v)$ in terms of $f(x,y)$.

III.2b. Change of Basis

$$\begin{cases} f(0,0) = F(0,0) + F(0,1) + F(1,0) + F(1,1) \\ f(0,1) = F(0,0) + F(0,1) \\ f(1,0) = F(0,0) + F(1,0) \\ f(1,1) = F(0,0) + F(1,1) \end{cases}$$

$$\begin{cases} F(0,0) = 0.5(-f(0,0) + f(0,1) + f(1,0) + f(1,1)) \\ F(0,1) = 0.5(f(0,0) + f(0,1) - f(1,0) - f(1,1)) \\ F(1,0) = 0.5(f(0,0) - f(0,1) + f(1,0) - f(1,1)) \\ F(1,1) = 0.5(f(0,0) - f(0,1) - f(1,0) + f(1,1)) \end{cases}$$

III.2c. Change of Basis

$$\begin{bmatrix} f(0,0) \\ f(0,1) \\ f(1,0) \\ f(1,1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F(0,0) \\ F(0,1) \\ F(1,0) \\ F(1,1) \end{bmatrix}$$

inverse of

$$\begin{bmatrix} F(0,0) \\ F(0,1) \\ F(1,0) \\ F(1,1) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} f(0,0) \\ f(0,1) \\ f(1,0) \\ f(1,1) \end{bmatrix}$$

III.2d. Change of Basis

$$f = F(0,0)\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{K_{0,0}} + F(0,1)\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{K_{0,1}} + F(1,0)\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}}_{K_{1,0}} + F(1,1)\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{K_{1,1}}$$

$$F = f(0,0)\underbrace{\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}}_{\mathbf{k}_{0,0}} + f(0,1)\underbrace{\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & -1/2 \end{bmatrix}}_{\mathbf{k}_{0,1}} + f(1,0)\underbrace{\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix}}_{\mathbf{k}_{1,0}} + f(1,1)\underbrace{\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}}_{\mathbf{k}_{1,1}}$$

III.3a. DIP Terminology

original image

$f = \Sigma_u \Sigma_v F(u,v) K_{u,v}$ → they define the **inverse transformation kernel**

$$f(x,y) = \Sigma_u \Sigma_v F(u,v)\, K_{u,v}(x,y)$$

transformed image

$F = \Sigma_x \Sigma_y f(x,y) k_{x,y}$ → they define the **forward transformation kernel**

$$F(u,v) = \Sigma_x \Sigma_y f(x,y)\, k_{x,y}(u,v)$$

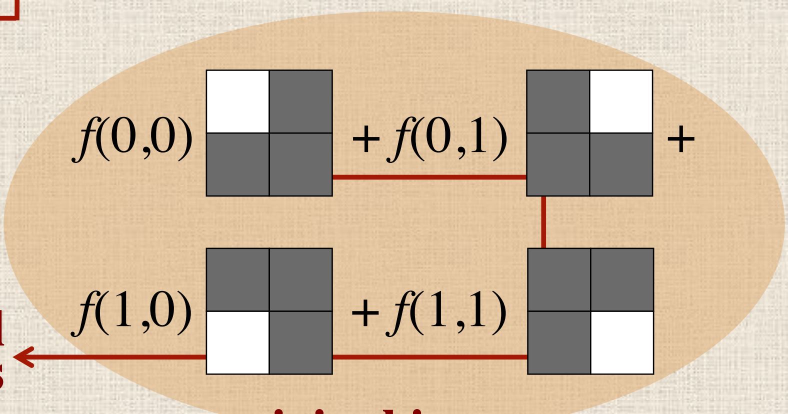
III.3b. DIP Terminology

Canonical basis:

$$c_{0,0} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ c_{0,1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$c_{1,0} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ c_{1,1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

canonical basis functions



canonical basis images

original image

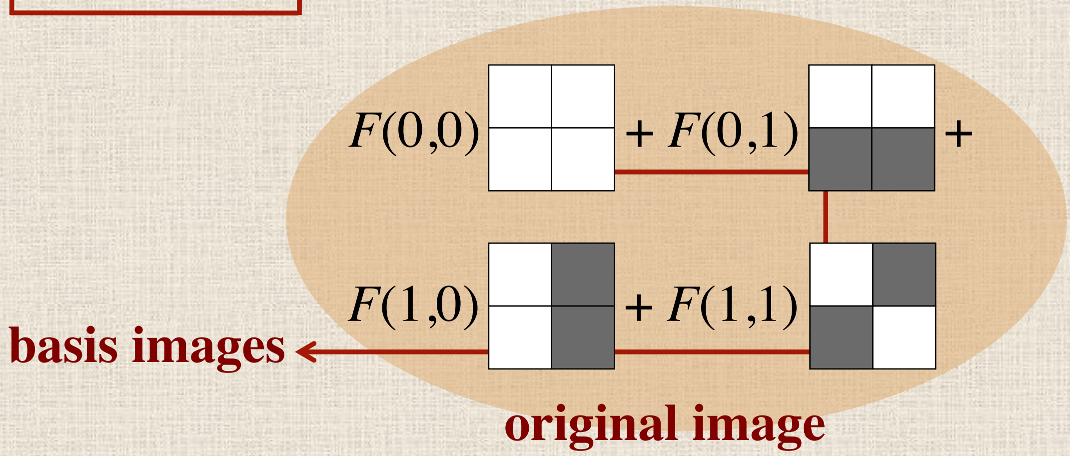
III.3c. DIP Terminology

Inverse transformation kernel:

basis functions

$$K_{0,0}=\begin{bmatrix}1&1\\1&1\end{bmatrix},\ K_{0,1}=\begin{bmatrix}1&1\\0&0\end{bmatrix}$$

$$K_{1,0}=\begin{bmatrix}1&0\\1&0\end{bmatrix},\ K_{1,1}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

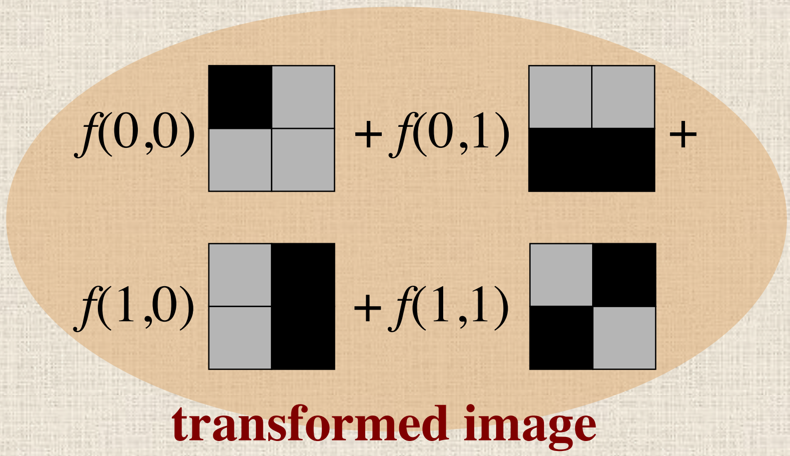


III.3d. DIP Terminology

Forward transformation kernel:

$$k_{0,0}=\begin{bmatrix}-1/2&1/2\\1/2&1/2\end{bmatrix},\ k_{0,1}=\begin{bmatrix}1/2&1/2\\-1/2&-1/2\end{bmatrix}$$

$$k_{1,0}=\begin{bmatrix}1/2&-1/2\\1/2&-1/2\end{bmatrix},\ k_{1,1}=\begin{bmatrix}1/2&-1/2\\-1/2&1/2\end{bmatrix}$$



III.4a. Generalization and Examples

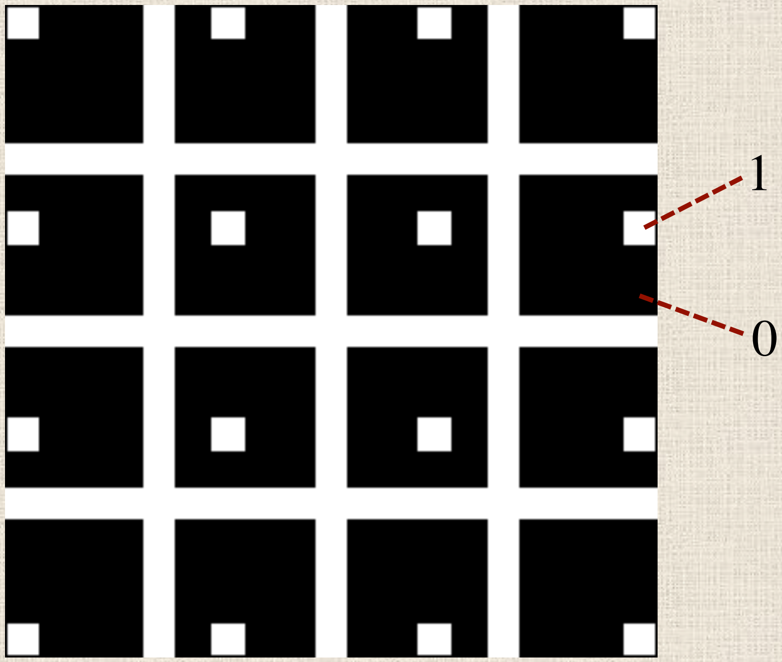
$$f = \sum_u \sum_v F(u,v)\, K_{u,v}$$
$$F = \sum_x \sum_y f(x,y)\, k_{x,y}$$

III.4b. Generalization and Examples

$$f = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v)\, K_{u,v}$$
$$F = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, k_{x,y}$$

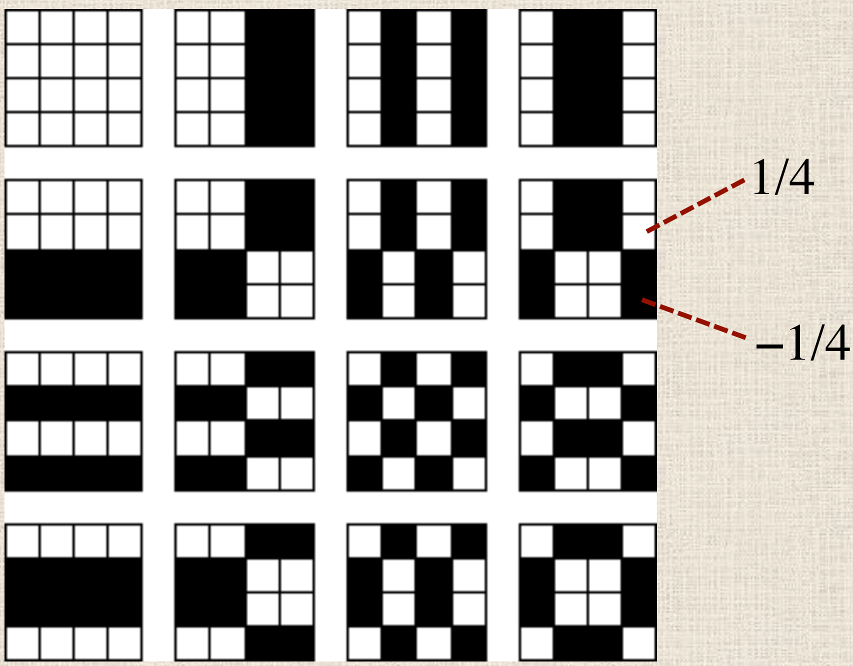
III.4c. Generalization and Examples

4×4 canonical basis images:



III.4d. Generalization and Examples

4×4 **Walsh** basis images:

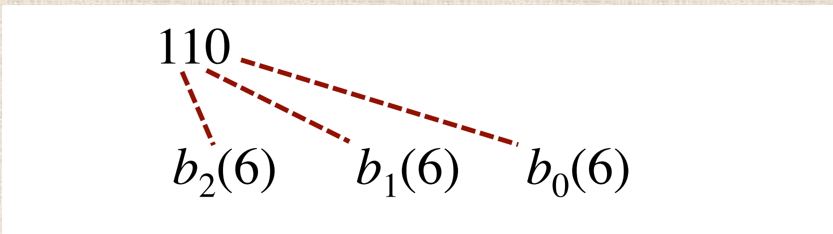


III.4e. Generalization and Examples

$2^n \times 2^n$ **Walsh** basis functions:

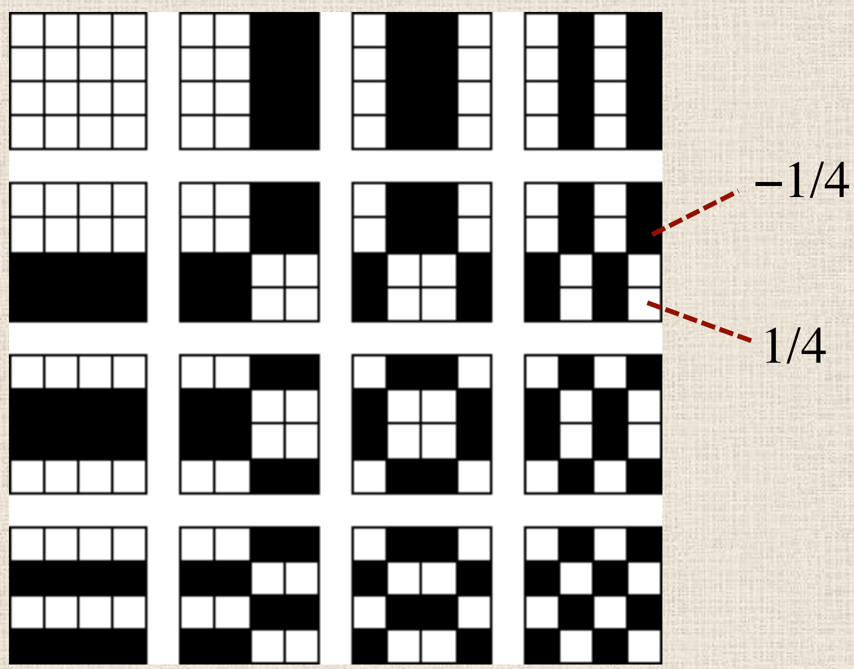
$$K_{u,v}(x,y) \doteq \frac{1}{2^n} \prod_{i=0}^{n-1} (-1)^{[b_i(x)b_{n-1-i}(u)+b_i(y)b_{n-1-i}(v)]}$$

where $b_i(x)$ is the i^{th} bit in the binary representation of x.



III.4f. Generalization and Examples

4×4 **Hadamard** basis images:

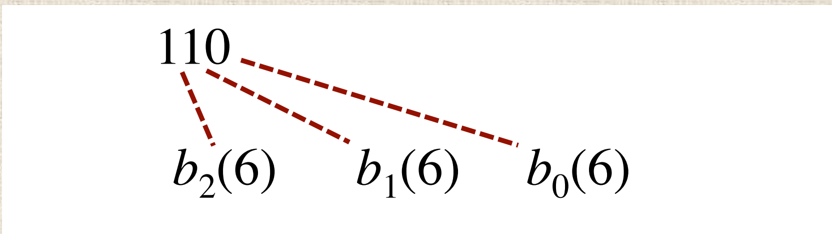


III.4g. Generalization and Examples

$2^n \times 2^n$ **Hadamard** basis functions:

$$K_{u,v}(x,y) \doteq \frac{1}{2^n}(-1)^{\sum_{i=0}^{n-1}[b_i(x)b_i(u)+b_i(y)b_i(v)]}$$

where $b_i(x)$ is the i^{th} bit in the binary representation of x.



III.4h. Generalization and Examples

$M \times N$ **Fourier** basis functions:

$$K_{u,v}(x,y) \doteq \frac{1}{MN} e^{2j\pi(ux/M + vy/N)}$$

$$= \frac{1}{M} e^{2j\pi ux/M} \frac{1}{N} e^{2j\pi vy/N}$$

$$= \underbrace{K_u^M(x) K_v^N(y)}$$

The kernel K is **separable**.

IV. The 2-D DFT: Definitions and Properties

IV.1a. Definitions

M and ***N*** $\doteq$ positive integers

function $\doteq$ total function from $(0.. M-1) \times (0.. N-1)$ into $\mathbb{C}$.

IV.1b. Definitions

For any u and x in $0..M-1$,
for any v and y in $0..N-1$:

$$K_{u,v}(x,y) \doteq \frac{1}{MN} e^{2j\pi(ux/M+vy/N)}$$

K is the **inverse Fourier transformation kernel**.

$$k_{x,y}(u,v) \doteq e^{-2j\pi(ux/M+vy/N)}$$

k is the **forward Fourier transformation kernel**.

IV.1c. Definitions

Consider a function F.
The function $f = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) K_{u,v}$
is the **inverse Fourier transform of** F.
It is denoted by $\mathcal{F}^{-1}(F)$.

Consider a function f.
The function $F = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) k_{x,y}$
is the **(forward) Fourier transform of** f.
It is denoted by $\mathcal{F}(f)$.

$\mathcal{F}$ is the **Fourier transform**.

IV.2. Justification

Consider two functions f and F.

$$\mathcal{F}^{-1}(\mathcal{F}(f)) = f$$

(i.e., if $F = \mathcal{F}(f)$ then $f = \mathcal{F}^{-1}(F)$)

$$\mathcal{F}(\mathcal{F}^{-1}(F)) = F$$

(i.e., if $f = \mathcal{F}^{-1}(F)$ then $F = \mathcal{F}(f)$)

IV.3a. Basic Properties

Consider a function F:

$$\mathcal{F}^{-1}(F) = \frac{1}{MN}\mathcal{F}(F^*)^*$$

IV.3b. Basic Properties

$\mathcal{F}$ is **linear**:

for any functions f_1 and f_2,
for any complex number c,
$\mathcal{F}(c f_1) = c\mathcal{F}(f_1)$ and $\mathcal{F}(f_1+f_2) = \mathcal{F}(f_1)+\mathcal{F}(f_2)$.

IV.3c. Basic Properties

Let f be a function.

$$\hat{F} \left| \begin{array}{l} \mathbb{Z}^2 \to \mathbb{C} \\ (u,v) \mapsto \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-2j\pi(ux/M + vy/N)} \end{array} \right.$$

is M and N **periodic**:

$$\forall (u,v) \in \mathbb{Z}^2,\ \hat{F}(u+M,v) = \hat{F}(u,v+N) = \hat{F}(u,v)$$

Moreover, **if f is a real function then**

$$\forall (u,v) \in \mathbb{Z}^2,\ \hat{F}(-u,-v) = \hat{F}^*(u,v)$$

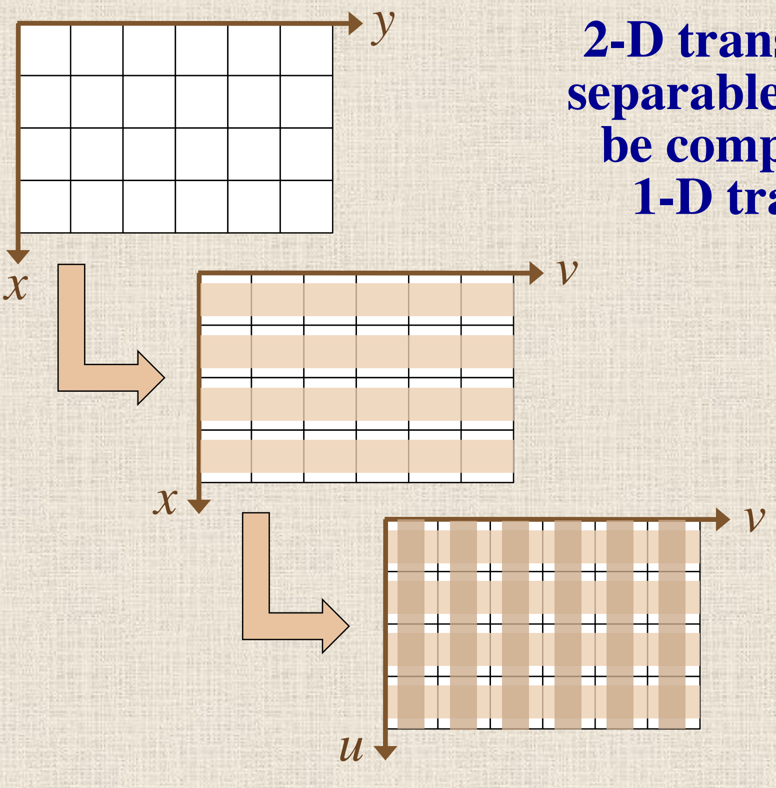
IV.3d. Basic Properties

Let f be a function and let F be its FT.
Let u_0 and v_0 be two integers.

The FT of $(x,y) \mapsto f(x,y)e^{2j\pi(u_0 x/M + v_0 y/N)}$ **is**
$(u,v) \mapsto \hat{F}(u-u_0, v-v_0)$

If *M* and *N* are even,
the FT of $(x,y) \mapsto (-1)^{x+y} f(x,y)$ **is**
$(u,v) \mapsto \hat{F}(u-M/2, v-N/2)$

IV.4a. Separability



2-D transforms with separable kernels can be computed using 1-D transforms.

IV.4b. Separability

$$F \doteq \mathcal{F}^{M,N}(f)$$

$$F(u,v) = \sum_{x=0}^{M-1}\left(\sum_{y=0}^{N-1} f(x,y)\, k_y^N(v)\right) k_x^M(u)$$

$$f(x,\bullet) \doteq y \mapsto f(x,y)$$

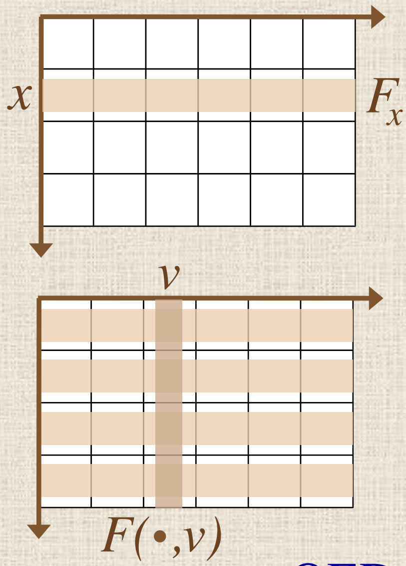
$$F_x \doteq \mathcal{F}^N(f(x,\bullet))$$

$$F(u,v) = \sum_{x=0}^{M-1} F_x(v)\, k_x^M(u)$$

$$F_\bullet(v) \doteq x \mapsto F_x(v)$$

$$F(\bullet,v) \doteq \mathcal{F}^M(F_\bullet(v))$$

$$F(u,v) = F(\bullet,v)(u)$$



QED

V. Convolution Theorem

V.1a. Convolution

M and ***N*** $\doteq$ positive integers

f and ***g*** $\doteq$ total functions

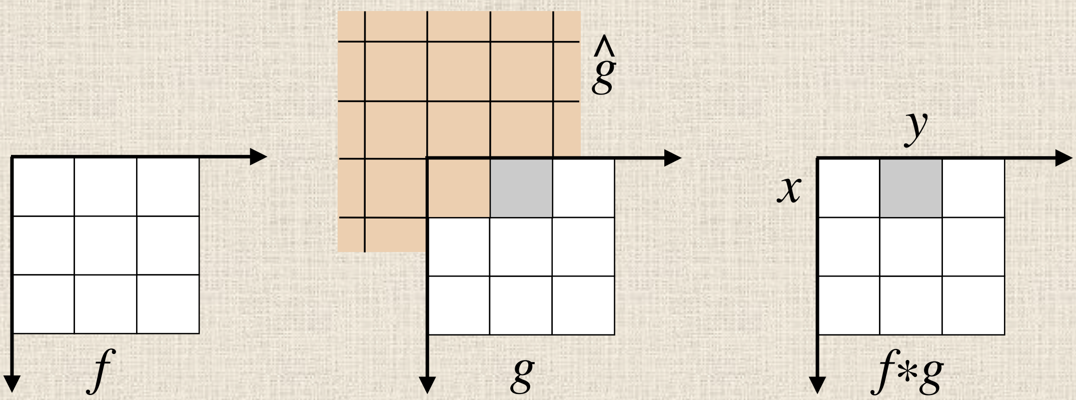
from $(0..\,M-1) \times (0..\,N-1)$ into $\mathbb{C}$

$\hat{g} \doteq M$ and N periodic extension of g to $\mathbb{Z}^2$

$$f*g \left| \begin{array}{l} (0..\,M-1) \times (0..\,N-1) \to \mathbb{C} \\ (x,y) \mapsto \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)\,\hat{g}(x-m,y-n) \end{array} \right.$$

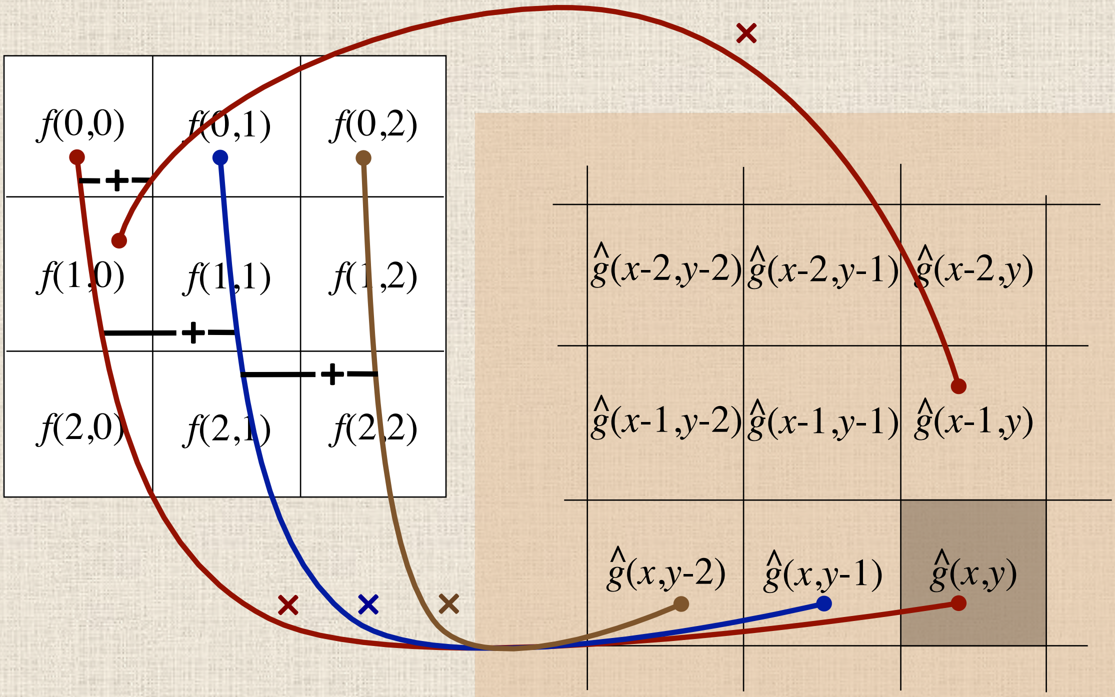
is the **convolution** of f with g.

V.1b. Convolution



$(f*g)(x,y)$ is a weighted sum of $\hat{g}$ values
from some neighborhood of (x,y) in $\hat{g}$;
the weights are the f values

V.1c. Convolution



V.1d. Convolution

Convolution is used in DIP for:

- blurring an image
- detecting (horizontal, vertical, diagonal) edges
- sharpening
- embossing
- ......

V.2. Theorem

(convolution / spatial domain) filter / kernel / mask

input image

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$$

$$f * g = \mathcal{F}^{-1}(\mathcal{F}(f)\mathcal{F}(g))$$

(frequency domain) filter / kernel / mask

output / filtered image

V.3a. Proof

$$\mathcal{F}(f * g)(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1}(f * g)(x,y)\,e^{-2j\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)\,\hat{g}(x-m,y-n)\,e^{-2j\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)\underbrace{\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\hat{g}(x-m,y-n)\,e^{-2j\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}}$$

V.3b. Proof

$$\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\hat{g}(x-m,y-n)e^{-2j\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$= e^{-2j\pi\left(\frac{um}{M}+\frac{vn}{N}\right)}\sum_{X=-m}^{M-1-m}\sum_{Y=-n}^{N-1-n}\hat{g}(X,Y)e^{-2j\pi\left(\frac{uX}{M}+\frac{vY}{N}\right)}$$

with X=x-m and Y=y-n

$$= e^{-2j\pi\left(\frac{um}{M}+\frac{vn}{N}\right)}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}g(x,y)e^{-2j\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

V.3c. Proof

$$\mathcal{F}(f*g)(u,v)$$

$$= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)e^{-2j\pi\left(\frac{um}{M}+\frac{vn}{N}\right)}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}g(x,y)e^{-2j\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$= \mathcal{F}(f)(u,v)\ \mathcal{F}(g)(u,v)$$

QED

VI. Fast FT (FFT)

VI.1. FT Computation

$M \doteq$ positive integer
$f \doteq$ total function from 0.. M-1 into $\mathbb{C}$

$$\mathcal{F}(f) \left| \begin{array}{l} 0..\, M\text{-}1 \rightarrow \mathbb{C} \\ \quad u \mapsto \sum_{x=0}^{M-1} f(x) e^{-2j\pi ux/M} \end{array} \right.$$

$add(M) \doteq$ number of (complex) additions
$mul(M) \doteq$ number of (complex) multiplications

VI.2. Brute-Force

$$\mathcal{F}(f) \;\Big|\; \begin{array}{l} 0..\,M\text{-}1 \rightarrow \mathbb{C} \\ \quad u \mapsto \sum_{x=0}^{M-1} f(x) e^{-2j\pi ux/M} \end{array}$$

is computed in $\mathcal{O}(M^2)$ time: $add(M)$ is $\mathcal{O}(M^2)$
$mul(M)$ is $\mathcal{O}(M^2)$

VI.3a. Successive-Doubling Method

$$\mathbf{F} \doteq \mathcal{F}(f)$$

$$\mathbf{W_M} \doteq e^{-2j\pi/M}$$

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-2j\pi ux/M} = \sum_{x=0}^{M-1} f(x) W_M^{ux}$$

Assume M is even

$$\mathbf{M} \doteq 2N$$

$$F(u) = \underbrace{\sum_{x=0}^{N-1} f(2x) W_N^{ux} + \sum_{x=0}^{N-1} f(2x+1) W_N^{ux} W_{2N}^{u}}$$

VI.3b. Successive-Doubling Method

$$\overbrace{F(u) = \sum_{x=0}^{N-1} f(2x) W_N^{ux} + \sum_{x=0}^{N-1} f(2x+1) W_N^{ux} W_{2N}^{u}}$$

For u in $0..N$-1

$$F(u) = \sum_{x=0}^{N-1} f(2x) W_N^{ux} + \sum_{x=0}^{N-1} f(2x+1) W_N^{ux} W_{2N}^{u}$$

$$F(u+N) = \sum_{x=0}^{N-1} f(2x) W_N^{ux} - \sum_{x=0}^{N-1} f(2x+1) W_N^{ux} W_{2N}^{u}$$

VI.3c. Successive-Doubling Method

$$\mathbf{F}_{even} \doteq \mathcal{F}(x \mapsto f(2x))$$

$$\mathbf{F}_{odd} \doteq \mathcal{F}(x \mapsto f(2x+1))$$

For u in $0..N$-1

$$F(u) = F_{even}(u) + F_{odd}(u) W_{2N}^{u}$$

$$F(u+N) = F_{even}(u) - F_{odd}(u) W_{2N}^{u}$$

$$add(2N) = 2\, add(N) + 2N$$

$$mul(2N) = 2\, mul(N) + N$$

VI.3d. Successive-Doubling Method

For any positive integer m

$add(2^m) = 2\, add(2^{m-1}) + 2^m$

$mul(2^m) = 2\, mul(2^{m-1}) + 2^{m-1}$

Moreover

$add(2^0) = 0$

$mul(2^0) = 0$

Therefore, for any M power of 2

$add(M) = M \log_2 M$

$mul(M) = \frac{1}{2} M \log_2 M$

VI.4a. Corollaries

$M \doteq$ positive integer

$f \doteq$ total function from $0..\, M{-}1$ into $\mathbb{C}$

$\mathcal{F}(f)$ can be computed in $\mathcal{O}(M \log M)$ time

VI.4b. Corollaries

$M \doteq$ positive integer
$F \doteq$ total function from $0..M-1$ into $\mathbb{C}$

$\mathcal{F}^{-1}(F)$ can be computed in $\mathcal{O}(M \log M)$ time using a forward FFT algorithm

VI.4c. Corollaries

M and $N \doteq$ positive integers
$f \doteq$ total function from $0..M-1 \times 0..N-1$ into $\mathbb{C}$

$\mathcal{F}(f)$ can be computed in $\mathcal{O}(MN \log MN)$ time using a forward 1-D FFT algorithm

VI.4d. Corollaries

M and $N \doteq$ positive integers
$F \doteq$ total function from $0..M-1 \times 0..N-1$ into $\mathbb{C}$

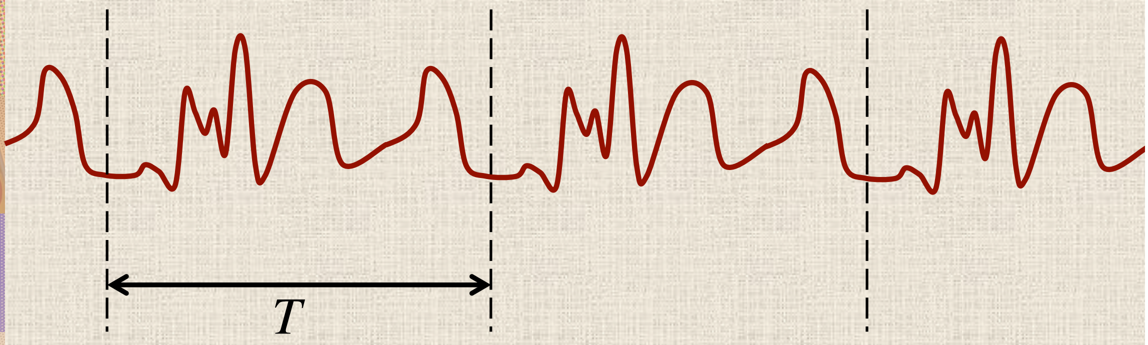
$\mathcal{F}^{-1}(F)$ can be computed in $\mathcal{O}(MN \log MN)$ time using a forward 1-D FFT algorithm

VII. Interpretation

VII.1a. Fourier Series

$T \doteq$ positive real number and $\omega \doteq 2\pi/T$

$f_a \doteq$ T-periodic total function from $\mathbb{R}$ into $[0,+\infty)$, differentiable and derivative continuous



VII.1b. Fourier Series

DEF PRO

There exist two unique* series

$(c_u)_{0..+\infty} \in [0,+\infty)^{0..+\infty}$ } **amplitude spectrum**

and $(\varphi_u)_{0..+\infty} \in [0,2\pi)^{0..+\infty}$ } **phase spectrum**

such that:

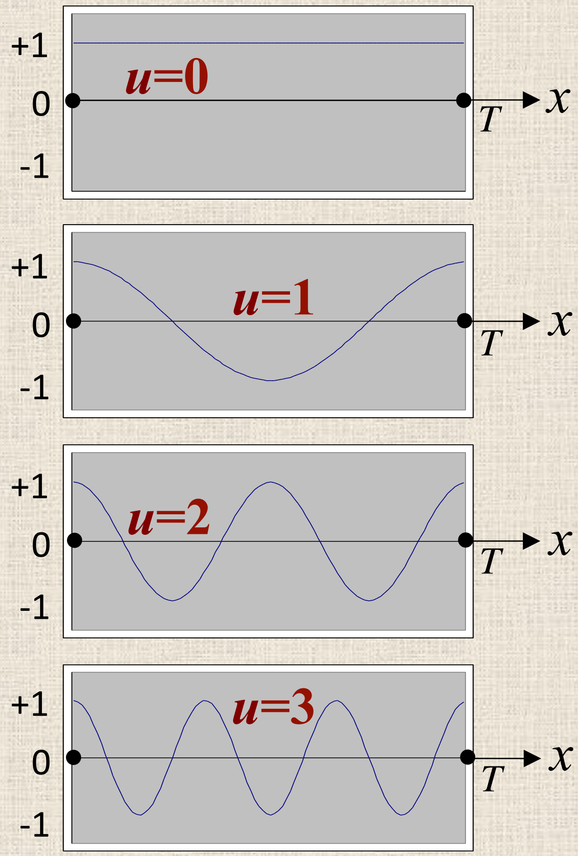
$$\forall x \in \mathbb{R},\ f_a(x) = \underbrace{\sum_{u=0}^{+\infty} c_u \cos(\omega u x + \varphi_u)}_{\text{Fourier series}}$$

Fourier series

VII.1c. Fourier Series

?

$x \mapsto \cos(\omega u x)$



VII.1d. Fourier Series

?

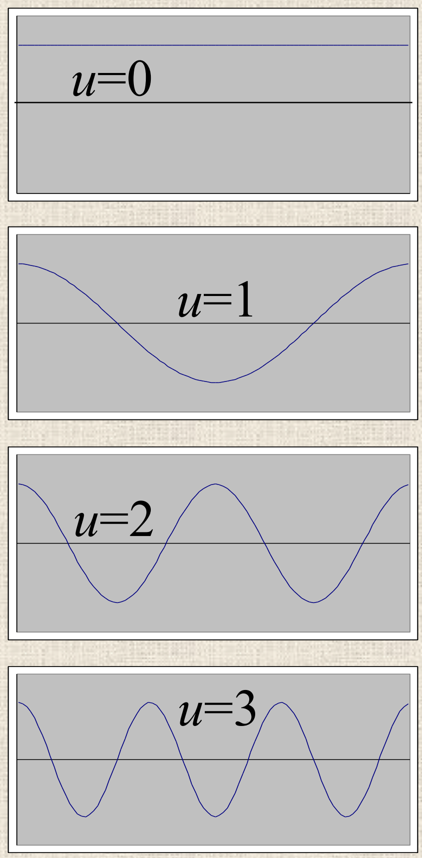
$x \mapsto \cos(\omega u x + \varphi_u)$

φ_0

φ_1

φ_2

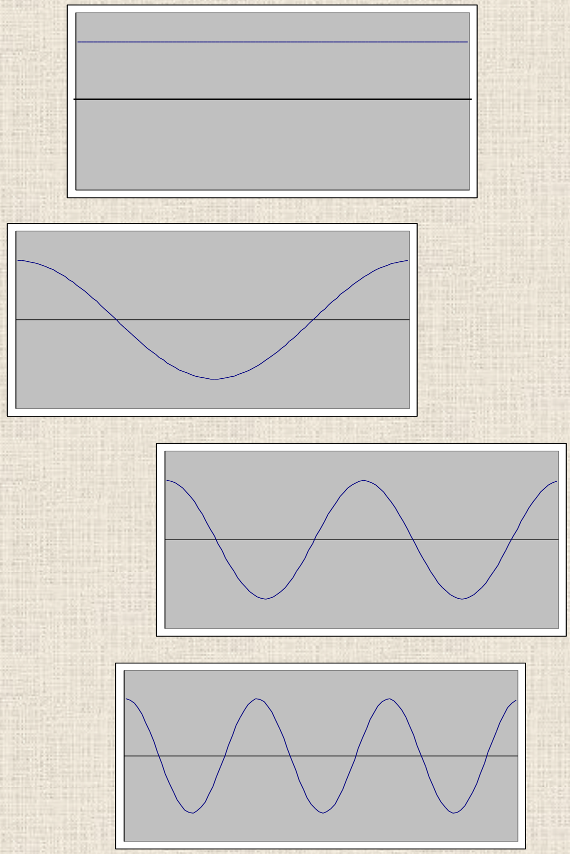
φ_3



VII.1e. Fourier Series

?

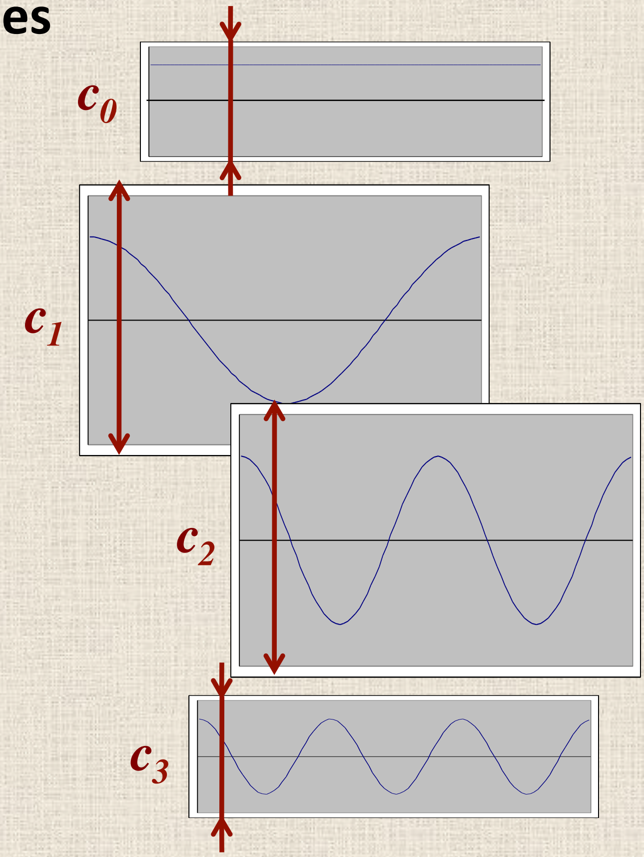
$$x \mapsto c_u \cos(\omega u x + \varphi_u)$$



VII.1f. Fourier Series

?

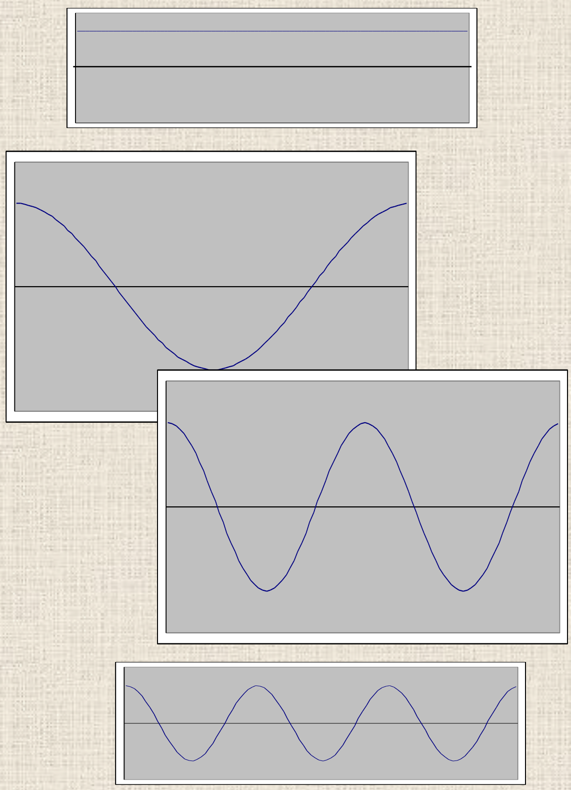
$$x \mapsto c_u \cos(\omega u x + \varphi_u)$$



VII.1g. Fourier Series

?

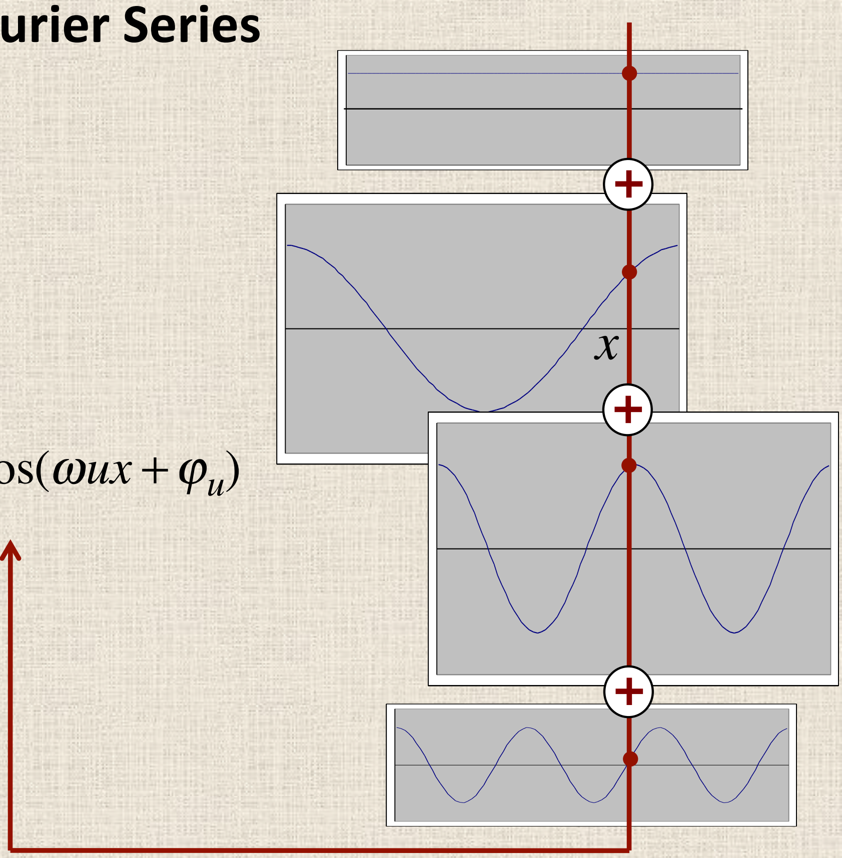
$$x \mapsto \sum_{u=0}^{+\infty} c_u \cos(\omega u x + \varphi_u)$$



VII.1h. Fourier Series

?

$$x \mapsto \sum_{u=0}^{+\infty} c_u \cos(\omega u x + \varphi_u)$$

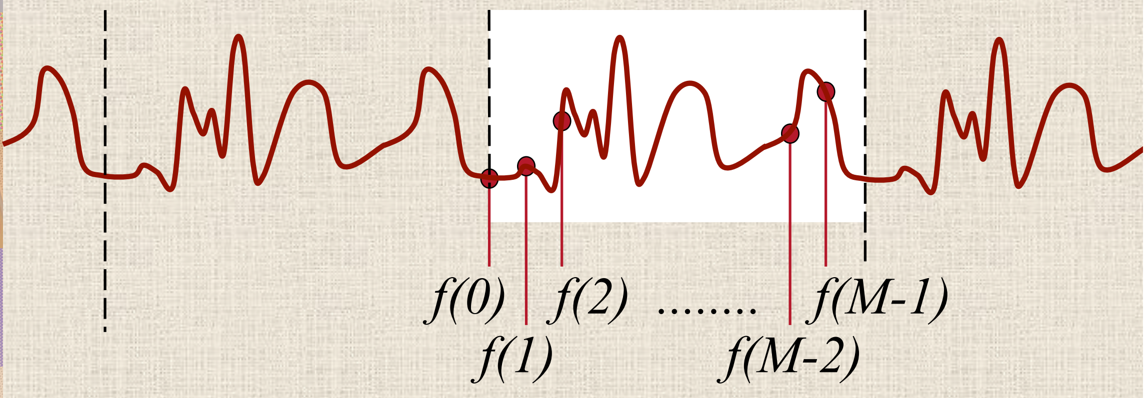


VII.2a. FT vs. Fourier Series

$M \doteq$ positive even integer and $K \doteq M/2$

$f \doteq$ total function from $0..\,M\text{-}1$ into $\mathbb{C}$,

$$\forall i \in 1..M\text{-}1,\ f(i) = f_a\left(\frac{T}{M}\, i\right)$$



VII.2b. FT vs. Fourier Series

$$f(x) = \frac{1}{M}\sum_{u=0}^{M-1} F(u)\, e^{2j\pi ux/M} \quad \text{where} \quad F \doteq \mathcal{F}(f)$$

$$= \boxed{\frac{1}{M} F(0)} = \boxed{\frac{1}{M}\sum_{X=0}^{M-1} f(X)}$$

$$+ \frac{1}{M}\sum_{u=1}^{K-1} F(u)\, e^{2j\pi ux/M}$$

group and arrange

$$+ \boxed{\frac{1}{M} F(K)(-1)^x} = \boxed{\frac{1}{M}(-1)^x \sum_{X=0}^{M-1} f(X)(-1)^X}$$

$$+ \boxed{\frac{1}{M}\sum_{u=K+1}^{2K-1} F(u)\, e^{2j\pi ux/M}} = \boxed{\frac{1}{M}\sum_{U=1}^{K-1} \left[F(U)\, e^{2j\pi Ux/M}\right]^*}$$

with $U = 2K - u$

VII.2c. FT vs. Fourier Series

$$f(x) = \sum_{u=0}^{K-1} C_u \cos(\Omega u x + \Phi_u) + \varepsilon(x)$$

$$\Omega = \frac{2\pi}{M}$$

$$\varepsilon(x) = \frac{(-1)^x}{M} \sum_{X=0}^{M-1} (-1)^X f(X)$$

$$\forall u \in 0..K\text{-}1,\ \Phi_u = \arg(F(u))$$

$$\begin{cases} C_0 = \dfrac{1}{M} \, | F(0) | \\ \forall u \in 1..K\text{-}1,\ C_u = \dfrac{2}{M} \, | F(u) | \end{cases}$$

VII.2d. FT vs. Fourier Series

$$f(x) = \sum_{u=0}^{K-1} C_u \cos(\Omega u x + \Phi_u) + \varepsilon(x)$$

$M \longrightarrow +\infty$

$$f_a(x) = \sum_{u=0}^{+\infty} c_u \cos(\omega u x + \varphi_u) \qquad \mathbf{0}$$

VII.3a. Forcing Periodicity

Interpretation FT/FS holds only if image periodic.

circular indexing

VII.3b. Forcing Periodicity

Interpretation FT/FS holds only if image periodic.

mismatch between the sides

⇩

FT is "biased"

VII.3c. Forcing Periodicity

Interpretation FT/FS holds only if image periodic.

reflected indexing

VII.3d. Forcing Periodicity

Interpretation FT/FS holds only if image periodic.

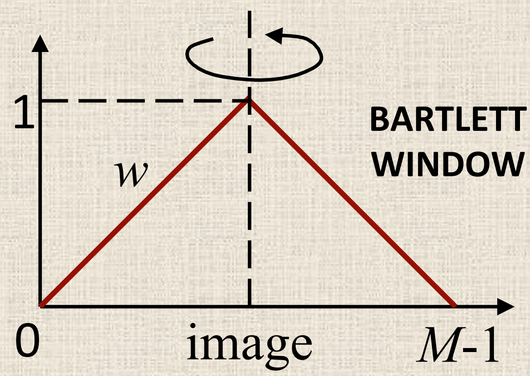
windowing

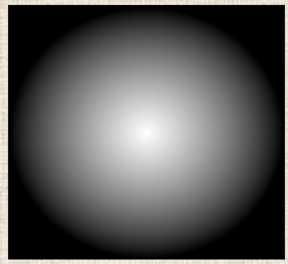
VII.3e. Forcing Periodicity

Compute $\mathcal{F}(wf)$ instead of $\mathcal{F}(f)$ with, e.g.,

$$w(x,y) = \begin{cases} 1 - \dfrac{\delta(x,y)}{K} & \text{if } \delta(x,y) \le K \\ 0 & \text{if } \delta(x,y) > K \end{cases}$$

and $\delta(x,y)$ distance between (x,y) and (K,K)

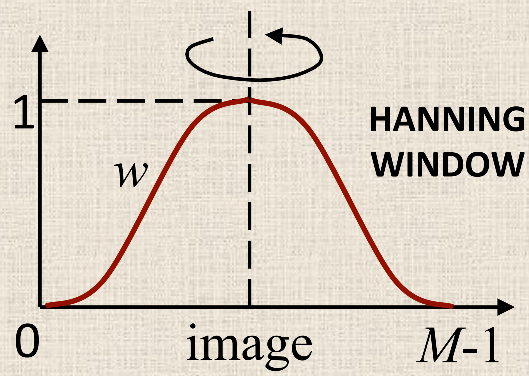


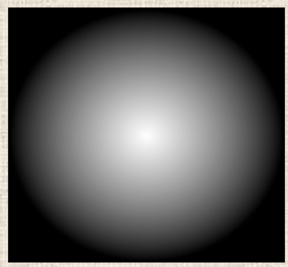


VII.3f. Forcing Periodicity

Compute $\mathcal{F}(wf)$ instead of $\mathcal{F}(f)$ with, e.g.,

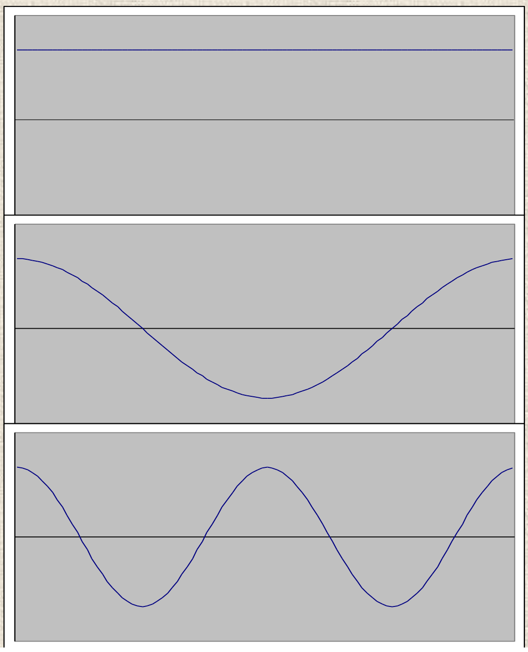
$$w(x,y) = \begin{cases} \dfrac{1}{2} - \dfrac{1}{2}\cos\left[\pi\left(1 - \dfrac{\delta(x,y)}{K}\right)\right] & \text{if } \delta(x,y) \le K \\ 0 & \text{if } \delta(x,y) > K \end{cases}$$





VII.4a. Shifting the Amplitude Spectrum

$$f(x) = \frac{1}{100} \sum_{u=0}^{99} F(u)\, e^{2j\pi ux/100}$$



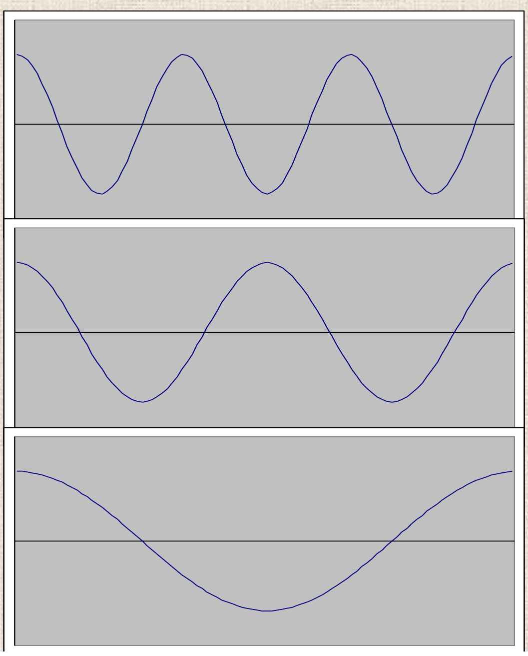
$|F(0)|$

$|F(1)|$

$|F(2)|$

VII.4b. Shifting the Amplitude Spectrum

$$f(x) = \frac{1}{100} \sum_{u=0}^{99} F(u)\, e^{2j\pi ux/100}$$



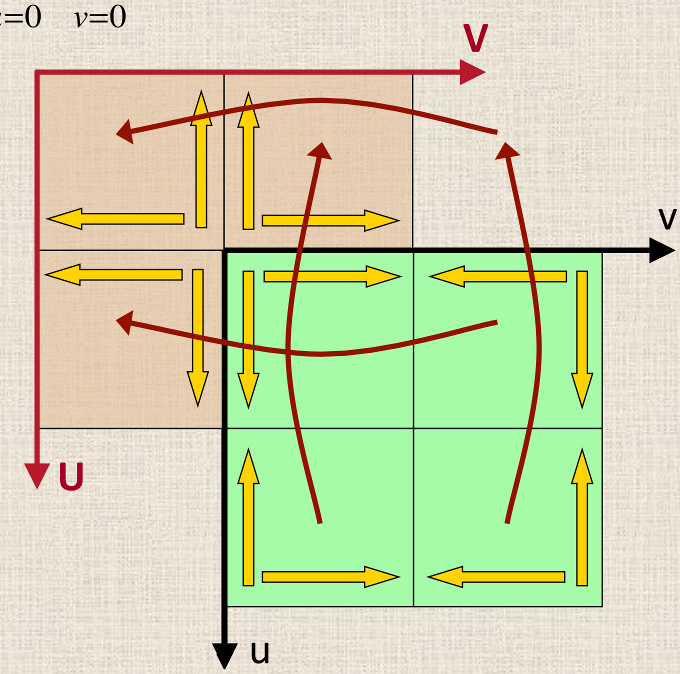
$|F(97)|=|F(3)|$

$|F(98)|=|F(2)|$

$|F(99)|=|F(1)|$

VII.4c. Shifting the Amplitude Spectrum

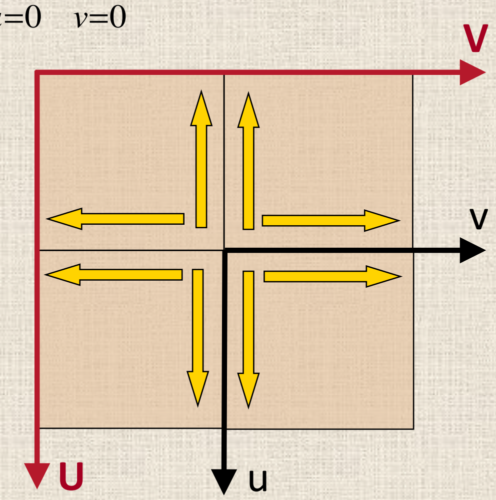
$$f(x,y) = \frac{1}{M^2} \sum_{u=0}^{M-1} \sum_{v=0}^{M-1} F(u,v)\, e^{2j\pi(ux+vy)/M}$$



frequency
increases
in that
direction

VII.4d. Shifting the Amplitude Spectrum

$$f(x,y) = \frac{1}{M^2} \sum_{u=0}^{M-1} \sum_{v=0}^{M-1} F(u,v)\, e^{2j\pi(ux+vy)/M}$$



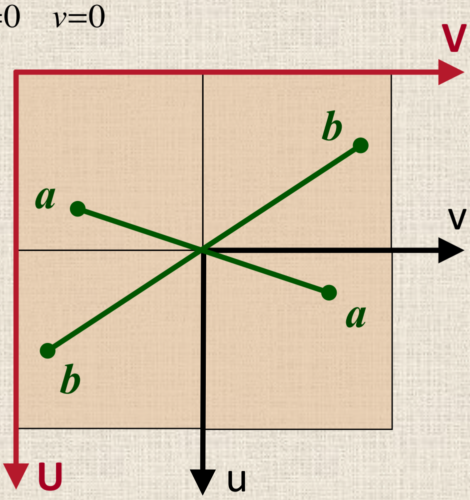
If we center the spectrum on (u,v)=(0,0) then frequency increases as we move in any direction away from that point.

VII.4e. Shifting the Amplitude Spectrum

$$f(x,y) = \frac{1}{M^2} \sum_{u=0}^{M-1} \sum_{v=0}^{M-1} F(u,v)\, e^{2j\pi(ux+vy)/M}$$

$$|\hat{F}(u,v)| = a \Rightarrow |\hat{F}(-u,-v)| = a$$

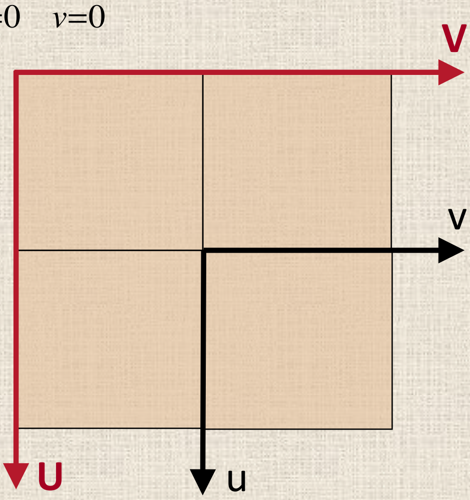
$$|\hat{F}(u,v)| = b \Rightarrow |\hat{F}(-u,-v)| = b$$



Moreover, the center of the shifted spectrum is center of symmetry.

VII.4f Shifting the Amplitude Spectrum

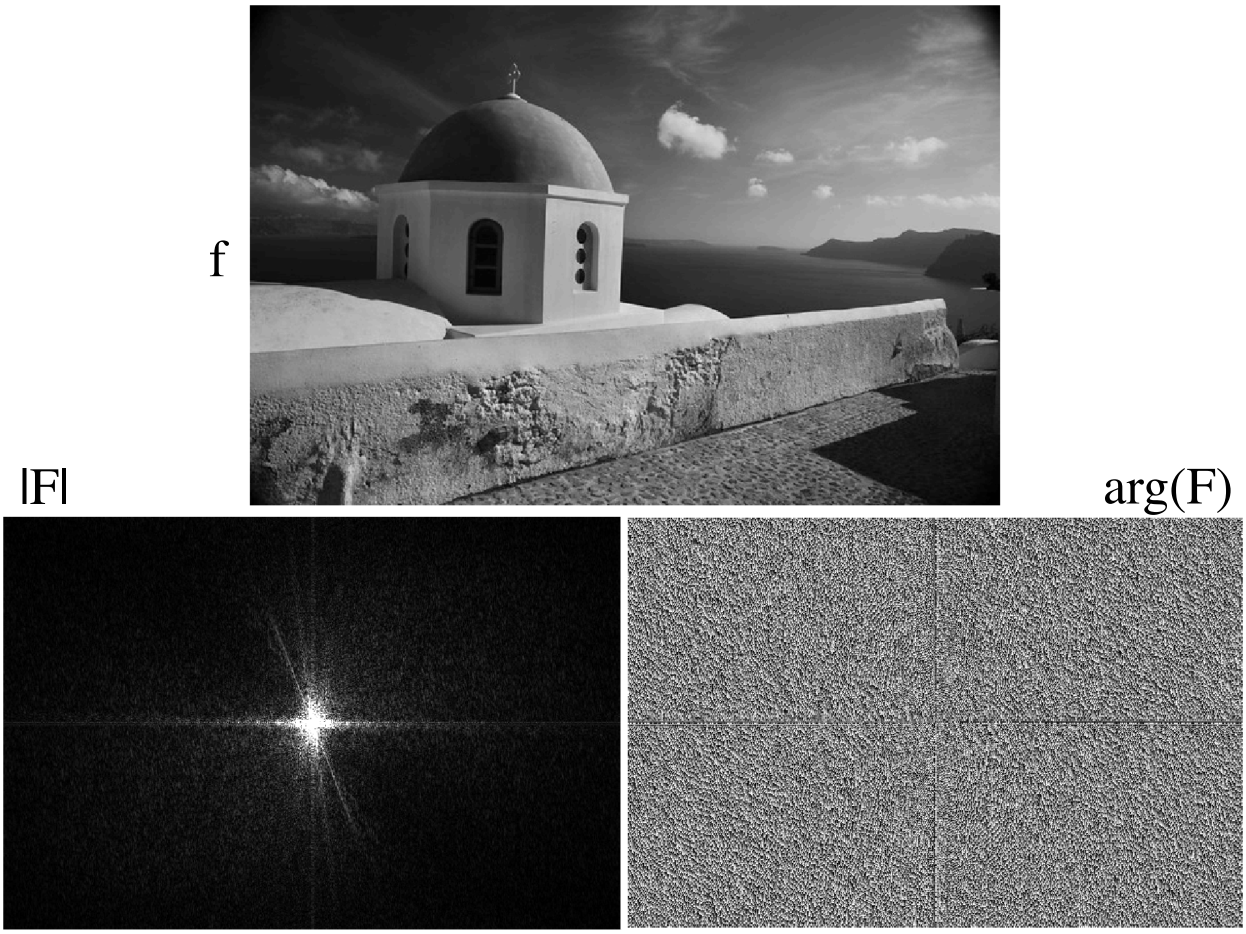
$$f(x,y) = \frac{1}{M^2} \sum_{u=0}^{M-1} \sum_{v=0}^{M-1} F(u,v)\, e^{2j\pi(ux+vy)/M}$$



CONCLUSION: Display $\hat{F}(u\text{-}K, v\text{-}K)$, i.e., replace f by $(x,y) \mapsto f(x,y)(-1)^{x+y}$

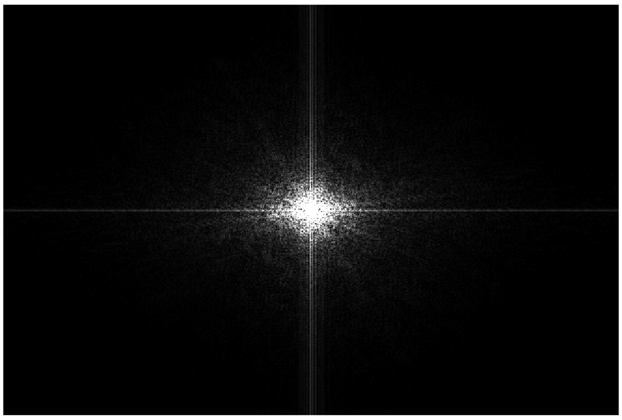
FT in the Field of DIP

Appendix

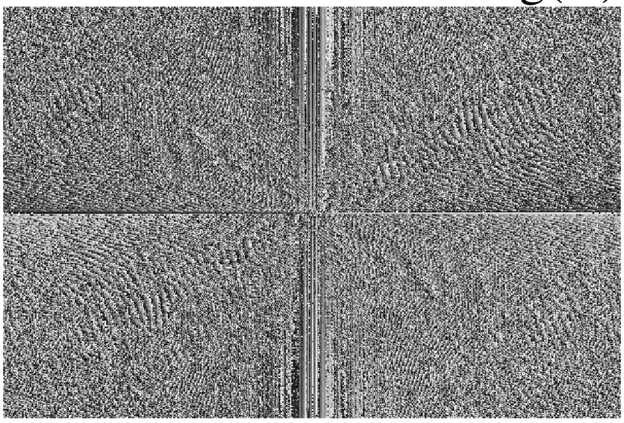


g

|G|



arg(G)



reconstructed
from |F| and arg(G)

reconstructed
from |G| and arg(F)