A Review of Complex Numbers

Commutativity, Associativity, Distributivity

A ***binary operation*** on a set S is a function from S^2 to S.

Binary operations are often denoted by symbols like ★
instead of letters like f, and we write u★v instead of ★((u,v)).

Let ★ and ❖ be two total binary operations on a nonempty set S:

★ is ***commutative*** iff $\forall(u,v)\in S^2$, u★v=v★u
★ is ***associative*** iff $\forall(u,v,w)\in S^3$, u★(v★w)=(u★v)★w
★ is ***distributive*** over ❖ iff $\forall(u,v,w)\in S^3$, [u★(v❖w)=(u★v)❖(u★w) ∧ (v❖w)★u=(v★u)❖(w★u)]

+ and × (addition and multiplication on $\mathbb{R}$)
are commutative and associative;
× is distributive over +.

Neutral Element, Inverse 3

Let ★ be a total binary operation on a nonempty set S.

n ∈ S is a ***neutral element*** for ★ iff: $\forall u \in S$, u★n=n★u=u

If there is a neutral element for ★ it is unique.

0 is the neutral element for + and 1 is the neutral element for ×.

Assume n is the neutral element for ★ and u is an element of S.
The element v of S is an ***inverse*** of u under ★ iff: u★v=v★u=n

Assume ★ is associative. If there is an inverse of u under ★ it is unique.

−2 is the inverse of 2 under + and 0.5 is the inverse of 2 under ×.

Fields 4

A ***field*** is a triple (S,★,❖) such that:

S is a nonempty set.

★ is a total binary operation on S.
There is a neutral element m for ★.
Every element of S has an inverse under ★.
★ is associative.
★ is commutative.

❖ is a total binary operation on S.
There is a neutral element n≠m for ❖.
Every element of S−{m} has an inverse under ❖.
❖ is associative.
❖ is commutative.

❖ is distributive over ★.

($\mathbb{R}$,+,×) is a field.

The Field of Complex Numbers 5

Consider the binary operations below:

$+ : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$
$((a,b),(c,d)) \mapsto (a+c, b+d)$

$\times : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$
$((a,b),(c,d)) \mapsto (ac-bd, bc+ad)$

$(\mathbb{R}^2,+,\times)$ is a field.

It is the field of ***complex numbers***.
$\mathbb{C}=\mathbb{R}^2$ is the set of complex numbers.

(0,0) is the neutral element for +.
The inverse under + of any element (a,b) of $\mathbb{C}$ is $(-a,-b)$.

This inverse is the ***opposite*** of (a,b).

(1,0) is the neutral element for $\times$.
The inverse under $\times$ of any $(a,b)\neq(0,0)$ of $\mathbb{C}$ is $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

This inverse is the ***reciprocal*** of (a,b).

Subtraction, Division, Exponentiation 6

Consider the field $(\mathbb{C},+,\times)$ of complex numbers.
Let z, z_1 and z_2 be complex numbers, and let n be a positive integer.

$\mathbf{-z}$ denotes the opposite of z.
$\mathbf{z_1-z_2}$ denotes $z_1+(-z_2)$.

$\mathbf{z^{-1}}$ denotes the reciprocal of z.
$\mathbf{z_1z_2}$ denotes $z_1\times z_2$.
$\mathbf{z_1/z_2}$ denotes $z_1\times(z_2^{-1})$.

$\mathbf{z^n}$ denotes $\overbrace{z\times z\times\ldots\times z}^{n \text{ times}}$.
$\mathbf{z^0}$ denotes $z\times(z^{-1})$.
$\mathbf{z^{-n}}$ denotes $(z^{-1})^n$.

Please Excuse My Dear Aunt Sally 7

arentheses > xponents > ultiplication = ivision > ddition = ubtraction

ultiplication = ivision: left to right

ddition = ubtraction: left to right

$z_1 \times z_2{}^n = z_1 \times (z_2{}^n)$

$z_1 \times z_2 / z_3 = (z_1 \times z_2) / z_3$
$z_1 / z_2 \times z_3 = (z_1 / z_2) \times z_3$

$z_1 + z_2 \times z_3 = z_1 + (z_2 \times z_3)$
$z_1 / z_2 - z_3 = (z_1 / z_2) - z_3$

$z_1 + z_2 - z_3 = (z_1 + z_2) - z_3$
$z_1 - z_2 + z_3 = (z_1 - z_2) + z_3$

$\mathbb{R}$ vs. $\mathbb{C}$, and j 8

Consider the field $(\mathbb{C},+,\times)$ of complex numbers.

Since $\forall a \in \mathbb{R}, \forall c \in \mathbb{R}, (a,0)+(c,0)=(a+c,0)$
and $\forall a \in \mathbb{R}, \forall c \in \mathbb{R}, (a,0)(c,0)=(ac,0)$,
any real number r can be seen as a complex number (r,0)
and we can write $\mathbb{R} \subset \mathbb{C}$.

Let **j**=(0,1).
j is the ***imaginary unit***.
We have $j^2=-1$.

Rectangular Form

$\forall z \in \mathbb{C}, \exists^1 a \in \mathbb{R}, \exists^1 b \in \mathbb{R}, z=a+jb$

a is the ***real part*** of z: $a=\mathcal{R}(z)$
b is the ***imaginary part*** of z: $b=\mathcal{I}(z)$
If $\mathcal{R}(z)=0$ then z is an ***imaginary number***.
a+jb is the ***rectangular form*** of z.

Let z_1 and z_2 be complex numbers.
$\mathcal{R}(z_1+z_2)=\mathcal{R}(z_1)+\mathcal{R}(z_2)$
$\mathcal{I}(z_1+z_2)=\mathcal{I}(z_1)+\mathcal{I}(z_2)$

Polar Form

$\forall z \in \mathbb{C}-\{0\}, \exists^1 r \in \mathbb{R}^+, \exists^1 \theta \in]-\pi,\pi], z=r(\cos\theta+j\sin\theta)$

r is the ***modulus*** of z: $r=|z|$
θ is the ***argument*** of z: $\theta=\arg(z)$
By convention, $|0|=0$ and $\arg(0)=0$.
$r(\cos\theta+j\sin\theta)$ is the ***polar form*** of z.

Let z_1 and z_2 be complex numbers.
$|z_1z_2|=|z_1||z_2|$
$\arg(z_1z_2) \equiv \arg(z_1)+\arg(z_2) \quad [2\pi]$

Conjugate 11

Let z be a complex number.

The ***conjugate*** of z is the complex number z***** defined by:
$\mathcal{R}(z^*)=\mathcal{R}(z)$ and $\mathcal{I}(z^*)=-\mathcal{I}(z)$.

Let z, z_1 and z_2 be complex numbers.

$\mathcal{R}(z)=(z+z^*)/2$	$\lvert z^*\rvert=\lvert z\rvert$
$\mathcal{I}(z)=(z-z^*)/(2j)$	$\arg(z^*) \equiv -\arg(z)\ [2\pi]$
$zz^*=\lvert z\rvert^2$	$(z_1+z_2)^*=z_1^*+z_2^*$
$z^{**}=z$	$(z_1z_2)^*=z_1^*z_2^*$

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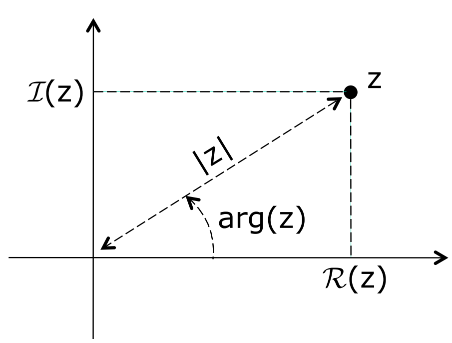
Complex Plane 12

Let z be a complex number.

z can be seen as a point in a 2-D plane (the ***complex plane***):
$\mathcal{R}(z)$ and $\mathcal{I}(z)$ are its rectangular coordinates, while
$\lvert z\rvert$ and arg(z) are its polar coordinates.



$\lvert z\rvert = \sqrt{\mathcal{R}(z)^2 + \mathcal{I}(z)^2}$
$\arg(z)=\mathrm{atan2}(\mathcal{I}(z),\mathcal{R}(z))$

$\mathcal{R}(z)=\lvert z\rvert\cos(\arg(z))$
$\mathcal{I}(z)=\lvert z\rvert\sin(\arg(z))$

$z \mapsto z+b$: translation
$z \mapsto az$ with $\lvert a\rvert=1$: rotation
$z \mapsto az$ with $a\in\mathbb{R}^+$: scaling
$z \mapsto z^*$: reflection

De Moivre’s and Euler’s Formulas

De Moivre’s formula

For any integer n and any nonzero complex number z:

$|z^n|=|z|^n$ and $\arg(z^n) \equiv n \arg(z)\ [2\pi]$

Euler’s formula

For any real number θ:

$\cos\theta + j\sin\theta = e^{j\theta}$

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END