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A Review of Vectors

Commutativity and Associativity

A ***binary operation*** on a set S is a function from S^2 to S.

Let ★ be a total binary operation on a nonempty set S:

★ is ***commutative*** iff $\forall (u, v) \in S^2$, u★v=v★u
★ is ***associative*** iff $\forall (u, v, w) \in S^3$, u★(v★w)=(u★v)★w

+ and × (addition and multiplication on $\mathbb{R}$)
are commutative and associative.

Neutral Element and Inverse

Let ★ be a total binary operation on a nonempty set S.

$n \in S$ is a ***neutral element*** for ★ iff: $\forall u \in S$, $u \star n = n \star u = u$

If there is a neutral element for ★ it is unique.

0 is the neutral element for + and 1 is the neutral element for ×.

Assume n is the neutral element for ★ and u is an element of S.
The element v of S is an ***inverse*** of u under ★ iff: $u \star v = v \star u = n$

Assume ★ is associative. If there is an inverse of u under ★ it is unique.

−2 is the inverse of 2 under + and 0.5 is the inverse of 2 under ×.

Vector Space: Definition

A ***vector space*** is a triple (V,+, **.**) such that:

☐ V is a nonempty set

☐ + is a total binary operation on V
+ is commutative and associative
there is a neutral element for +
for any v of V, there is an inverse of v under +

☐ **.** is a total function from $\mathbb{R} \times V$ to V
$\forall v \in V$, $1.v = v$
$\forall (a,b) \in \mathbb{R}^2$, $\forall v \in V$, $(ab).v = a.(b.v)$
$\forall (a,b) \in \mathbb{R}^2$, $\forall (u,v) \in V^2$, $[(a+b).v = a.v + b.v \wedge a.(u+v) = a.u + a.v]$

CAREFUL
The same symbol + is used here to denote two different binary operations. The symbol + in (a+b).v denotes the addition on $\mathbb{R}$, i.e., a binary operation on $\mathbb{R}$. The symbol + in a.(u+v) denotes a binary operation on V; moreover, it is assumed that **.** has precedence over +.

Vector Space: Example 5

Let + be the binary operation on $\mathbb{R}^3$ defined by:
$(x,y,z)+(x',y',z')=(x+x',y+y',z+z')$
Let **.** be the function from $\mathbb{R}\times\mathbb{R}^3$ to $\mathbb{R}^3$ defined by:
$a.(x,y,z)=(ax,ay,az)$
$(\mathbb{R}^3,+,\,.)$ is a vector space.

Basic Terminology 6

Let (V,+, **.**) be a vector space.

- Elements of V are ***vectors***
  Elements of $\mathbb{R}$ are ***scalars***

- + is the ***vector addition***
  u+v reads "the sum of u and v"
  The neutral element for + is the ***zero vector*** and denoted by **0**
  The inverse of v under + is the ***opposite*** of v and denoted by **−v**
  −u+v means (−u)+v and u−v means u+(−v)

- **.** is the ***scalar multiplication***
  a**.**v reads "the scalar multiplication of v by a"

Basic Properties 7

Let $(V,+,.)$ be a vector space.

$\forall v \in V,\ 0.v=0$
$\forall a \in \mathbb{R},\ a.0=0$
$\forall a \in \mathbb{R},\ \forall v \in V,\ [a.v=0 \rightarrow (a=0 \lor v=0)]$
$\forall v \in V,\ (-1).v=-v$

Bases and Subspaces 8

Let $(V,+,.)$ be a vector space,
let n be a positive integer and let $w=(v_1,v_2,\ldots,v_n)$ be an element of V^n.

- ☐ w is ***linearly independent*** iff:
  $\forall (a_1,a_2,\ldots,a_n) \in \mathbb{R}^n,\ [a_1.v_1+a_2.v_2+\ldots+a_n.v_n=0 \rightarrow a_1=a_2=\ldots=a_n=0]$.
  Note that $a_1.v_1+a_2.v_2+\ldots+a_n.v_n$ is a ***linear combination*** of w.

- ☐ w ***generates*** $\{v \in V \mid \exists (a_1,a_2,\ldots,a_n) \in \mathbb{R}^n,\ v=a_1.v_1+a_2.v_2+\ldots+a_n.v_n\}$.
  This set is a vector space*. It is the ***subspace*** generated by w.

- ☐ w is a ***basis*** iff:
  w is linearly independent and generates V.

Consider the vector space $(\mathbb{R}^3,+,.)$ as in slide 5.

((2,0,2),(−1,3,5),(1,3,7)) is linearly dependent.
((2,0,2),(−1,3,5)) is linearly independent but is not a basis.
((2,0,2),(−1,3,5),(1,4,−1)) is a basis.
((1,0,0),(0,1,0),(0,0,1)) is another basis.

Space Dimension and Vector Coordinates

Let $(V,+,\boldsymbol{.})$ be a vector space
and let $w=(v_1,v_2,\ldots,v_n)$ be a basis.

All bases of V are n-tuples.
n is the ***dimension*** of the vector space.

$\forall v \in V,\ \exists!\,(a_1,a_2,\ldots,a_n)\in\mathbb{R}^n,\ v=a_1\boldsymbol{.}v_1+a_2\boldsymbol{.}v_2+\ldots+a_n\boldsymbol{.}v_n$
$(a_1,a_2,\ldots,a_n)$ is the tuple of ***coordinates*** of v relative to w.

Consider the vector space $(\mathbb{R}^3,+,\boldsymbol{.})$ as in slide 5. It is of dimension 3.
Consider the vectors $i=(1,0,0)$, $j=(0,1,0)$, $k=(0,0,1)$,
and $v_1=(2,0,2)$, $v_2=(-1,3,5)$, $v_3=(1,4,-1)$.

$(0,1,0)$ is the tuple of coordinates of v_2 relative to the basis (v_1,v_2,v_3).
$(-1,3,5)$ is the tuple of coordinates of v_2 relative to the basis (i,j,k).

Coordinates of Sums and Products

Let $(V,+,\boldsymbol{.})$ be a vector space
and let $w=(v_1,v_2,\ldots,v_n)$ be a basis.

Consider a vector v.
Assume $(a_1,a_2,\ldots,a_n)$ is the tuple of coordinates of v relative to w.
We may then write $v(a_1,a_2,\ldots,a_n)$.

Consider a real number λ, two vectors $u(a_1,a_2,\ldots,a_n)$ and $v(b_1,b_2,\ldots,b_n)$.
We have $(u+v)(a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$ and $(\lambda\boldsymbol{.}u)(\lambda a_1, \lambda a_2, \ldots, \lambda a_n)$

Inner Product

Consider a vector space $(V,+,\centerdot)$.
A total function $\langle.,.\rangle$ from V^2 to $\mathbb{R}$ is an ***inner product*** iff:

- ☐ $\forall v \in V$, $[(v=0 \rightarrow \langle v,v\rangle=0) \wedge (v\neq 0 \rightarrow \langle v,v\rangle>0)]$
- ☐ $\forall (u,v) \in V^2$, $\langle u,v\rangle=\langle v,u\rangle$
- ☐ $\forall a \in \mathbb{R}$, $\forall (u,v,w) \in V^3$, $[\langle a\centerdot u,w\rangle=a\langle u,w\rangle \wedge \langle u+v,w\rangle=\langle u,w\rangle+\langle v,w\rangle]$

Consider a vector space $(V,+,\centerdot)$ and an inner product $\langle.,.\rangle$.
$(V,+,\centerdot,\langle.,.\rangle)$ is a ***Euclidean vector space***.

Consider the vector space $(\mathbb{R}^3,+,\centerdot)$ as in slide 5
and the function $\langle.,.\rangle$ from $\mathbb{R}^3\times\mathbb{R}^3$ to $\mathbb{R}$ defined by:
$\langle(x,y,z),(x',y',z')\rangle=xx'+yy'+zz'$

$(\mathbb{R}^3,+,\centerdot,\langle.,.\rangle)$ is a Euclidean vector space.

Orthogonal Vectors

Consider a Euclidean vector space $(V,+,\centerdot,\langle.,.\rangle)$.
Two vectors u and v are ***orthogonal vectors*** iff $\langle u,v\rangle=0$.
A basis $(v_1,v_2,\dots,v_n)$ is an ***orthogonal basis*** iff it is a basis
whose vectors are pairwise orthogonal.

Consider the Euclidean vector space $(\mathbb{R}^3,+,\centerdot,\langle.,.\rangle)$ as in slide 11.
Consider the vectors $i=(1,0,0)$, $j=(0,1,0)$, $k=(0,0,1)$,
and $v_1=(2,0,2)$, $v_2=(-1,3,5)$, $v_3=(1,4,-1)$.

(i,j,k) is an orthogonal basis, (v_1,v_2,v_3) is not.

Norm 13

Consider a vector space (V,+, .).
A total function |.| from V to $\mathbb{R}$ is a ***norm*** iff:

- ☐ $\forall v \in V$, $[(v=0 \rightarrow |v|=0) \wedge (v \neq 0 \rightarrow |v|>0)]$
- ☐ $\forall (u,v) \in V^2$, $|u+v| \leq |u|+|v|$
- ☐ $\forall a \in \mathbb{R}$, $\forall v \in V$, $|a.v|=|a||v|$

Consider a vector space (V,+, .) and a norm |.|.
(V,+, . , |.|) is a ***normed vector space***.

Euclidean Norm 14

Let (V,+, . , <.,.>) be a Euclidean vector space and
let |.| be the function from V to $\mathbb{R}$ defined by: $\forall v \in V$, $|v|=\sqrt{\langle v,v\rangle}$.

> |.| is a norm.
> It is the ***Euclidean norm***.

A vector v is a ***unit vector*** iff |v|=1.
A basis $(v_1,v_2,\ldots,v_n)$ is an ***orthonormal basis*** iff
it is an orthogonal basis whose vectors are unit vectors.

Consider an orthonormal basis $(v_1,v_2,\ldots,v_n)$.

> $\langle u(a_1,a_2,\ldots,a_n),v(b_1,b_2,\ldots,b_n)\rangle=a_1b_1+a_2b_2+\ldots+a_nb_n$
> $|u(a_1,a_2,\ldots,a_n)|=\sqrt{(a_1^2+a_2^2+\ldots+a_n^2)}$

Consider two nonzero vectors u and v. The element θ of [0,π]
such that $\cos(\theta)=\langle u,v\rangle/(|u||v|)$ is the ***angle*** between u and v.

Distance 15

Consider a set S.
A total function d from S^2 to $\mathbb{R}$ is a ***distance*** on S iff:

☐ $\forall(u,v)\in S^2$, $[(u=v \rightarrow d(u,v)=0) \land (u\neq v \rightarrow d(u,v)>0)]$
☐ $\forall(u,v,w)\in S^3$, $d(u,w)\leq d(u,v)+d(v,w)$
☐ $\forall(u,v)\in S^2$, $d(u,v)=d(v,u)$

Consider a set S and a distance d on S.
(S,d) is a ***metric space***.

Euclidean Distance 16

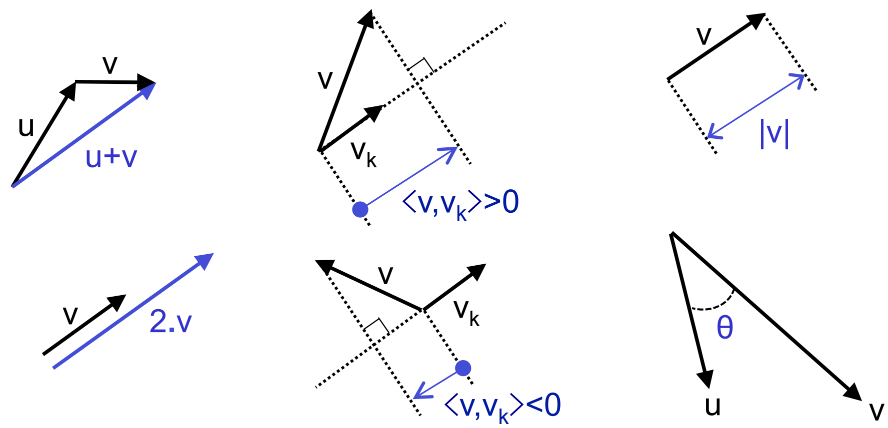
Let $(V,+,\cdot,\langle .,. \rangle)$ be a Euclidean vector space and
let d be the function from V^2 to $\mathbb{R}$ defined by: $\forall(u,v)\in V^2$, $d(u,v)=|u-v|$

d is a distance on V.
It is the ***Euclidean distance***.

Consider an orthonormal basis $(v_1,v_2,\ldots,v_n)$.

$d(u(a_1,a_2,\ldots,a_n),v(b_1,b_2,\ldots,b_n))=\sqrt{[(a_1-b_1)^2+(a_2-b_2)^2+\ldots+(a_n-b_n)^2]}$

Graphical Representation 17



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