A PASCAL-LIKE BOUND FOR THE NUMBER OF NECKLACES WITH FIXED DENSITY

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ABSTRACT. A bound resembling Pascal's identity is presented for binary necklaces with fixed density using Lyndon words with fixed density. The result is generalized to k-ary necklaces and Lyndon words with fixed content. The bound arises in the study of Nichols algebras of diagonal type.

1. INTRODUCTION

A *necklace* is the lexicographically smallest word in an equivalence class of words under rotation[1]. A *Lyndon word* is a primitive necklace, which means that it is strictly smaller than any of its non-trivial rotations. The *density* of a binary word is the number of 1s it contains. Let $\mathbf{N}(n,d)$ denote the set of all binary necklaces of length n and density d. Similarly, let $\mathbf{L}(n,d)$ denote the set of all binary Lyndon words of length n and density d. Let the cardinality of these two sets be denoted by $N(n,d)$ and $L(n,d)$, respectively. The following formulae are well-known for any $n \geq 1$ and $0 \leq d \leq n$ (see [4] and [10, Sect. 2]):

$$N(n,d) = \frac{1}{n} \sum_{j|\gcd(n,d)} \phi(j) \binom{n/j}{d/j}, \qquad L(n,d) = \frac{1}{n} \sum_{j|\gcd(n,d)} \mu(j) \binom{n/j}{d/j},$$

where ϕ and μ denote Euler's totient function and the Möbius function, respectively. Note that $N(n,d) = N(n,n-d)$ and $L(n,d) = L(n,n-d)$. Our main result is to prove the following Pascal's identity-like bound on $N(n,d)$.

Theorem 1.1. *For any* $0 < d < n$,

$$N(n,d) \leq L(n-1,d) + L(n-1,d-1).$$

Bounds on necklaces and Lyndon words like the one presented in the above theorem are generally difficult to prove directly from their enumeration formulae. Previous bounds on these objects used in algorithm analysis [10] use a combinatorial-style proof, and that is the approach we follow in this paper. To prove this theorem, we actually show something stronger. Let $\Sigma_k =$

[1]A necklace is often thought to be an equivalence class of words under rotation. Here we use the term to identify each such class with its *minimal word* [7].

$\{0, 1, 2, \ldots, k-1\}$ denote an alphabet of size k. Let $\mathbf{W}_k(n_0, n_1, \ldots, n_{k-1})$ denote the set of all words over Σ_k where each symbol i appears precisely n_i times. Such a set is said to be a set with *fixed content*, as used by [1, Sect. 18.3.3]. In a similar manner let $\mathbf{N}_k(n_0, n_1, \ldots, n_{k-1})$ denote the set of necklaces with the given fixed content and let $\mathbf{L}_k(n_0, n_1, \ldots, n_{k-1})$ denote the set of Lyndon words with the given fixed content. Let the cardinality of these two sets be denoted by $N_k(n_0, n_1, \ldots, n_{k-1})$ and $L_k(n_0, n_1, \ldots, n_{k-1})$, respectively. In [4] and [10, Sect. 2], explicit formulas for the number of necklaces and Lyndon words with fixed content are given:

$$N_k(n_0, n_1, \ldots, n_{k-1}) = \frac{1}{n} \sum_{j \mid \gcd(n_0, \ldots, n_{k-1})} \phi(j) \frac{(n/j)!}{(n_0/j)! \cdots (n_{k-1}/j)!},$$

$$L_k(n_0, n_1, \ldots, n_{k-1}) = \frac{1}{n} \sum_{j \mid \gcd(n_0, \ldots, n_{k-1})} \mu(j) \frac{(n/j)!}{(n_0/j)! \cdots (n_{k-1}/j)!},$$

where $n = n_0 + n_1 + \cdots + n_{k-1}$. In Section 3, we prove the following more general result.

Theorem 1.2. *Let $k \geq 2$ and $n_0, n_1, \ldots, n_{k-1} \geq 1$ be positive integers. Then*

$$N_k(n_0, \ldots, n_{k-1}) \leq \sum_{i=0}^{k-1} L_k(n_0, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_{k-1}).$$

Moreover, the inequality is strict if $k > 2$.

Observe that when $k = 2$, Theorem 1.2 simplifies to Theorem 1.1. After presenting some preliminary materials in Section 2, we prove the theorems in Section 3. For more background on Lyndon words see [6].

1.1. **An Application.** The inequalities in Theorems 1.1 and 1.2 seem to be new. They arised with the study of Nichols algebras of diagonal type in [5] in order to determine whether such a Nichols algebra is a free algebra. Roughly, the inequality implies that a certain rational function is in fact a polynomial, and freeness of the Nichols algebra holds if none of these polynomials vanish at the point of an affine space determined by the braiding of the Nichols algebra. The calculation of the zeros of such a polynomial simplifies significantly if the inequality is known to be strict. Strictness when $k = 2$ is further discussed in Section 4.

2. Background

A word is called a *prenecklace*, if it is the prefix of some necklace. For any non-empty word α let $|\alpha|$ denote the length of α and let $\mathrm{lyn}(\alpha)$ be the length of the longest prefix of α that is a Lyndon word. The following observation is applied implicitly in many necklace algorithms including the ones from [2]. It is an extension of the classical property stated by Duval [3, P. 380] that every necklace has the form β^t where β is a Lyndon word and $t \geq 1$.

Lemma 2.1. *Let $\alpha = a_1 a_2 \cdots a_n$ be a prenecklace with $p = \operatorname{lyn}(\alpha)$. Then α is a necklace if and only if $n \bmod p = 0$.*

Proof. Suppose α is a necklace. We know that α has the form β^t for some Lyndon word β and some $t \geq 1$. Since $p = \operatorname{lyn}(\alpha)$, $|\beta| \leq p$. If $|\beta| = p$ then we are done. Otherwise $\beta = a_1 a_2 \cdots a_q$ for some $q < p$. Thus $a_1 a_2 \cdots a_p = (a_1 a_2 \cdots a_q)^j a_1 a_2 \cdots a_i$ for some $j \geq 1$ and $1 \leq i < q$. However, since β is a Lyndon word, $a_1 a_2 \cdots a_q < a_{i+1} \cdots a_q a_1 a_2 \cdots a_i$. But this implies that $a_1 a_2 \cdots a_i (a_1 a_2 \cdots a_q)^j < a_1 a_2 \cdots a_p$, which contradicts $a_1 a_2 \cdots a_p$ being a Lyndon word. Now suppose $n \bmod p = 0$. Since $a_1 a_2 \cdots a_p$ is a Lyndon word, we clearly have that α is less than or equal to any of its rotations, and hence is a necklace. □

The following theorem was named the *Fundamental theorem of necklaces* by Ruskey and a proof of the theorem is presented in [9] which builds on a foundation of results from Duval [3]. There is a similar statement by Reutenauer [8] on page 164.

Theorem 2.2. [2, Thm. 2.1] *Let $\alpha = a_1 \cdots a_{n-1}$ be a prenecklace over the alphabet Σ_k, where $n, k \geq 2$. Let $p = \operatorname{lyn}(\alpha)$ and let $b \in \Sigma_k$. Then αb is a prenecklace if and only if $a_{n-p} \leq b$. In this case,*

$$\operatorname{lyn}(\alpha b) = \begin{cases} p & \text{if } b = a_{n-p}, \\ n & \text{if } b > a_{n-p}. \end{cases}$$

Corollary 2.3. *If $\alpha = a_1 a_2 \cdots a_n$ is a prenecklace and $b > a_n$, then the word $a_1 a_2 \cdots a_{n-1} b$ is a Lyndon word.*

Proof. Since α is a prenecklace, then so is $a_1 a_2 \cdots a_{n-1}$. From Theorem 2.2, we have $a_{n-p} \leq a_n$ since α is a prenecklace. Since $a_n < b$ then $a_{n-p} < b$ and thus Theorem 2.2 implies that $a_1 a_2 \cdots a_{n-1} b$ is a Lyndon word. □

Corollary 2.4. *If $\alpha = a_1 a_2 \cdots a_n$ is a necklace and $b > a_1$, then αb is a Lyndon word.*

Proof. Let $p = \operatorname{lyn}(\alpha)$. By Lemma 2.1, $n \bmod p = 0$. Since α is a prenecklace, Theorem 2.2 implies that αb is a Lyndon word since $b > a_1 = a_{n+1-p}$. □

The following result follows from inductively applying Theorem 2.2 and corresponds to [2, Lem. 2.3].

Corollary 2.5. *Let $\alpha = a_1a_2\cdots a_n$ be a prenecklace with $p = \mathrm{lyn}(\alpha) < n$, that is, α is not a Lyndon word. Then $\alpha = (a_1a_2\cdots a_p)^j a_1a_2\cdots a_i$ for some $j \geq 1$ and $1 \leq i \leq p$.*

The following remark follows directly from the definition of a necklace.

Remark 2.6. *Every necklace over Σ_k that contains a 0 and a non-0 element must begin with 0 and end with a non-0.*

3. Proof of Main Theorems

A necklace $\alpha = a_1a_2\cdots a_n$ is said to be *stable* if $a_1a_2\cdots a_{n-1}$ is a Lyndon word; otherwise α is *unstable*. We prove Theorem 1.2, which generalizes Theorem 1.1, by partitioning $\mathbf{N}_k(n_0, n_1, \ldots, n_{k-1})$ into two sets $\mathbf{S}$ and $\mathbf{U}$, which contain the stable and unstable necklaces of $\mathbf{N}_k(n_0, n_1, \ldots, n_{k-1})$, respectively.

Lemma 3.1. *Let $k \geq 2$ and $n_0, n_1, \ldots, n_{k-1} \geq 1$ be positive integers. Then*

$$|\mathbf{S}| = \sum_{i=1}^{k-1} L_k(n_0, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_{k-1}).$$

Proof. Since each $n_i > 0$, by Remark 2.6 every necklace in $\mathbf{S}$ must begin with 0 and end with a non-0. By partitioning $\mathbf{S}$ by its last symbol, the result follows from Corollary 2.4 and the definition of stable. □

It remains to show that $|\mathbf{U}| \leq L_k(n_0 - 1, n_1, n_2, \ldots, n_{k-1})$. We assume $k \geq 2$ and each $n_i \geq 1$. Let $\alpha = a_1a_2\cdots a_n$ be a necklace in $\mathbf{U}$. From Remark 2.6, $a_1 = 0$ and $a_n > 0$. Let $\alpha' = a_1a_2\cdots a_{n-1}$ and let $x = a_n$. Since α is unstable, α' is a prenecklace, but not a Lyndon word. Thus, applying Corollary 2.5, we can write α' as $(a_1a_2\cdots a_p)^j a_1a_2\cdots a_i$ where $j \geq 1$ and $1 \leq i \leq p$. For the upcoming function f, we define z, which we call the *index of α associated with f*, to be the largest index no more than i such that $a_z = 0$. Such an index exists since $a_1 = 0$. Thus

$$\alpha = (a_1a_2\cdots a_p)^j a_1a_2\cdots a_{z-1}a_z a_{z+1}a_{z+2}\cdots a_i x.$$

Consider the function $f : \mathbf{U} \to \mathbf{L}_k(n_0 - 1, n_1, n_2, \ldots, n_{k-1})$ as follows:

$$f(\alpha) = \begin{cases} (a_1a_2\cdots a_p)^j a_1a_2\cdots a_{z-1}x & \text{if } z = i; \\ (a_1a_2\cdots a_p)^j x a_i a_{i-1}\cdots a_{z+2}a_1a_2\cdots a_{z-1}a_{z+1} & \text{if } z < i. \end{cases}$$

Clearly $f(\alpha)$ has the required content. To see that $f(\alpha)$ is a Lyndon word, consider two cases depending on z. Suppose $z = i$. As stated earlier, $(a_1a_2\cdots a_p)^j a_1a_2\cdots a_z = \alpha'$ is a prenecklace. Thus since $x > a_z$, $(a_1a_2\cdots a_p)^j a_1a_2\cdots a_{z-1}x$ is a Lyndon word by Corollary 2.3. Suppose $z < i$. Note that $(a_1a_2\cdots a_p)^j$ is a necklace and $a_1 = 0$. Since each symbol in $xa_ia_{i-1}\cdots a_{z+2}$ is non-0, $\beta = (a_1a_2\cdots a_p)^j xa_ia_{i-1}\cdots a_{z+2}$ is a Lyndon word by Corollary 2.4. Observe that $\beta a_1 a_2 \cdots a_z$ is a prenecklace, since it is a prefix of the necklace β^2. Thus, since $a_{z+1} > 0$ from the definition of z and $a_z = 0$, Corollary 2.3 implies $\beta a_1 a_2 \cdots a_{z-1} a_{z+1}$ is a Lyndon word. Hence $f(\alpha)$ is a Lyndon word.

Lemma 3.2. *Let $k \geq 2$ and $n_0, n_1, \ldots, n_{k-1}$ be positive integers. Then*

$$|\mathbf{U}| \leq L_k(n_0-1, n_1, n_2, \ldots, n_{k-1}).$$

Proof. We prove that f is one-to-one. Consider necklaces α and β in $\mathbf{U}$. Set:

$$\begin{aligned}\alpha &= (a_1 a_2 \cdots a_p)^j a_1 a_2 \cdots a_{z-1} a_z a_{z+1} a_{z+2} \cdots a_i x,\\ \beta &= (b_1 b_2 \cdots b_{p'})^{j'} b_1 b_2 \cdots b_{z'-1} b_{z'} b_{z'+1} b_{z'+2} \cdots b_{i'} x',\end{aligned}$$

where z (resp. z') is the index of α (resp. of β) associated with f. Suppose $f(\alpha) = f(\beta)$. If $jp = j'p'$, then $p = p'$ since both $a_1 a_2 \cdots a_p$ and $b_1 b_2 \cdots b_{p'}$ are Lyndon words. If $p = p'$ then clearly $\alpha = \beta$ by the definition of f. This means that $p \neq j'p'$ and $p' \neq jp$. Thus, without loss of generality assume that $jp < j'p'$. Suppose $p < p' < jp$, which implies $j > 1$. But this implies there exists $1 \leq h < j$ and $0 < t \leq p$ such that $b_1 b_2 \cdots b_{p'} = (a_1 a_2 \cdots a_p)^h a_1 a_2 \cdots a_t$ which contradicts $b_1 b_2 \cdots b_{p'}$ being a Lyndon word. By a similar argument we cannot have $p' < p < j'p'$. Thus $jp < p'$, which by the definition of f, implies $j' = 1$ and $a_1 a_2 \cdots a_p = b_1 b_2 \cdots b_p$. Consider two cases based on $\ell = |a_{z+2} \cdots a_i x|$.

(i) Suppose $jp + \ell \geq p' > jp$. This implies $\ell > 0$. Let $\ell' = |b_{z'+2} \cdots b_{i'} x'|$. Recall $a_1 = b_1 = 0$ and each letter of $a_{z+1} \cdots a_i x$ and $b_{z'+1} \cdots b_{i'} x'$ is non-0. Thus, since $f(\alpha) = f(\beta)$, we must have $jp + \ell = p' + \ell'$, which means $z = z'$. But this means $x' = a_{i'+1} = b_{i'+1}$, which means that β ends with $b_1 b_2 \cdots b_{i'+1}$ which is a proper prefix of $b_1 b_2 \cdots b_{p'}$. Since $i' + 1 < p'$ we have $n \bmod p' \neq 0$ and thus Lemma 2.1 implies that β is not a necklace, a contradiction.

(ii) Suppose $jp + \ell < p'$. In this case, for $f(\alpha) = f(\beta)$, it must be that some suffix of $b_1 b_2 \cdots b_{p'}$ is a prefix of $a_1 a_2 \cdots a_{z-1} = b_1 b_2 \cdots b_{z-1}$ which contradicts $b_1 b_2 \cdots b_{p'}$ being a Lyndon word.

$\square$

Together Lemma 3.1 and Lemma 3.2 prove Theorem 1.1. To complete the proof of Theorem 1.2, the following lemma proves that for $k > 2$ the function f is not a bijection.

Lemma 3.3. *The function f is not a surjection when $k > 2$.*

Proof. We consider two cases depending on the parity of n_0. For each case we demonstrate a word γ in $\mathbf{L}_k(n_0-1, n_1, n_2, \ldots, n_{k-1})$ that is not in the range of f. Consider $\alpha \in \mathbf{U}$ and let $\gamma = f(\alpha)$. Observe that since $k > 2$, the shortest prefix u of $\gamma = uv$ containing two different symbols is also a prefix of α. Moreover, if v ends with $0^j t$, with $j \geq 0$ and $t \in \{1, 2, \ldots, k-1\}$, then $0^{j+1}t$ is a subword of α.

- Case 1: $n_0 = 2j+1$ for $j \geq 0$. There is no necklace in $\mathbf{U}$ that maps to the Lyndon word $0^j 1^{n_1} 2^{n_2} \cdots (k-1)^{n_{k-1}-1} 0^j (k-1)$ because such a necklace would have to start with $0^j 1$ but have the subword 0^{j+1}.
- Case 2: $n_0 = 2j$ for $j \geq 1$. There is no necklace in $\mathbf{U}$ that maps to the Lyndon word $0^j (k-1)^{n_{k-1}} \cdots 3^{n_3} 2^{n_2} 1^{n_1-1} 0^{j-1} 1$ because such a necklace would have to start with $0^j(k-1)$ but have the subword $0^j 1$.

□

4. Special Cases When $N(n,d) = L(n-1,d) + L(n-1,d-1)$

In this section we discuss when the inequality given by Theorem 1.1 is equality. Recall that the density d of a binary word is the number of 1s it contains.

Lemma 4.1. *If $d \in \{1, 2, n-2, n-1\}$ and $0 < d < n$ then $N(n,d) = L(n-1,d) + L(n-1,d-1)$ except for $(n,d) = (2,1)$.*

Proof. Recall that $N(n,d) = N(n,n-d)$ and $L(n,d) = L(n,n-d)$. Thus it suffices to prove the result for $d = 1$ and $d = 2$. For $d = 1$ and $n > 2$, $\mathbf{N}(n,1) = \{0^{n-1}1\}$, $\mathbf{L}(n-1,1) = \{0^{n-2}1\}$, and $\mathbf{L}(n-1,0) = \emptyset$, and thus the result holds. For $d = 2$, consider the parity of n. If n is even, then $\mathbf{L}(n,2) = \{0^{n-j-2}10^j1 \mid j \in \{0, 1, \ldots, \frac{n-2}{2} - 1\}\ \}$ and $\mathbf{N}(n,2) = \mathbf{L}(n,2) \cup \{0^{(n-2)/2}10^{(n-2)/2}1\}$. If n is odd, then $\mathbf{N}(n,d) = \mathbf{L}(n,2) = \{0^{n-j-2}10^j1 \mid j \in \{0, 1, \ldots, \frac{n-3}{2}\}\}$. Thus, $N(n,2) = \lfloor \frac{n}{2} \rfloor$, $L(n-1,2) = \lfloor \frac{n-2}{2} \rfloor$, and $L(n-1,1) = 1$, and therefore the result holds. □

We now consider the other values of d.

Lemma 4.2. *If $2 < d < n-2$ then $N(n,d) = L(n-1,d) + L(n-1,d-1)$ if and only if $(n,d) \in \{(6,3), (7,3), (7,4), (8,4), (9,3), (9,6)\}$.*

Proof. The claim can be verified directly from the enumeration formulae for $n \leq 10$. A table of relevant values for $N(n,d)$ (resp. $L(n,d)$) are provided in the left (resp. right) table below.

n/d	3	4	5	6	7
6	**4**				
7	**5**	**5**			
8	7	**10**	7		
9	**10**	14	14	**10**	
10	12	22	26	22	12

n/d	2	3	4	5	6	7
5	2	2				
6	2	3	2			
7	3	5	5	3		
8	3	7	8	7	3	
9	4	9	14	14	9	4

For $n > 10$, let $z = n{-}d$ (the number of 0s). Recall that $N(n,d) = N(n,n{-}d)$ and $L(n,d) = L(n,n{-}d)$. Thus, if $N(n,d) = L(n{-}1,d) + L(n{-}1,d{-}1)$ then $N(n,n{-}d) = L(n{-}1,n{-}d) + L(n{-}1,n{-}(d{-}1))$. Thus, we can assume $2 < z \leq n/2$. Consider three cases for $z = 3$, $z = 4$, and $z \geq 5$. Each result is proved by specifying a Lyndon word $\beta \in \mathbf{L}(n{-}1,d) = \mathbf{L}_2(z{-}1,d)$ not in the range of f when the domain is the set of unstable necklaces in $\mathbf{N}(n,d) = \mathbf{N}_2(z,d)$.

- Case: $z = 3$. Depending on the parity of n let β be either 01^t01^{t+1} or 01^t01^{t+2}. Since $n > 10$, we have $t \geq 3$. Consider any necklace α in $\mathbf{N}_2(3,d)$ which can be written as $01^{t_1}01^{t_2}01^{t_3}$ for non-negative integers t_1,t_2,t_3. Note that if $t_2 < t_1$ then the the rotation of α starting with 01^{t_2} is less than α, a contradiction to α being a necklace. Similarly for t_3. Thus, $t_2,t_3 \geq t_1$. But for any such (unstable) α, $f(\alpha) = 01^{t_1}01^{t_2+t_3} \neq \beta$.
- Case: $z = 4$. Depending on the value of (n mod 3) let β be one of the words $01^t01^t01^{t+1}$, $01^t01^{t+1}01^{t+1}$, and $01^t01^{t+1}01^{t+2}$. Since $n > 10$, we have $t \geq 2$. Consider any unstable necklace $\alpha = a_1a_2\cdots a_n$ in $\mathbf{N}_2(4,d)$ which can be written as $01^{t_1}01^{t_2}01^{t_3}01^{t_4}$ for non-negative integers t_1,t_2,t_3,t_4. Again, observe that if $t_2 < t_1$ then the rotation of α starting with 01^{t_2} is less than α, a contradiction to α being a necklace. Similarly for t_3,t_4. Thus, $t_2,t_3,t_4 \geq t_1$. Moreover, since $n > 10$, if $t_4 = 1$ then at least one of t_1 or t_2 or t_3 is greater than one which similarly contradicts α being a necklace as the rotation starting with 01^{t_4} will be less than α. Thus $t_4 > 1$. Recall we can write α as $(a_1a_2\cdots a_p)^j a_1a_2\cdots a_ix$ where $x = 1$, $j \geq 1$, and $1 \leq i \leq p$ where p is the length of the longest Lyndon prefix of $a_1a_2\cdots a_{n-1}$. Since clearly $a_1 = 0$ and $a_p = 1$, $p \in \{t_1{+}1, t_1{+}t_2{+}2, t_1{+}t_2{+}t_3{+}3\}$. If $p = t_1 + 1$ then $j = 3$ and $t_1 = t_2 = t_3$ by the definition of p. However then $f(\alpha) = 01^{t_1}01^{t_1}01^{t_1+t_4}$ which is not equal to any possible β. If $p = t_1{+}t_2{+}2$ then $t_1 = t_3$ and $t_2 > t_1$ by the definition of p, and thus $t_2 > t_3$. However, then $f(\alpha) = 01^{t_1}01^{t_2+t_4-1}01^{t_3+1}$ which is not equal to any possible β. Finally, if $p = t_1{+}t_2{+}t_3{+}3$ then $t_3 > t_1$ by the definition of p. However, then $f(\alpha) = 01^{t_1}01^{t_2}01^{t_3+t_4}$ which is also not equal to any possible β.

- Case: $z \geq 5$. Recall $n > 10$ and $z \leq n/2$. Consider the length $n-1$ word $\beta = 0011^s01(01)^t$ where $t = z-4$ and $s = n-2t-6$. Since $z \geq 5$, we have $t \geq 1$ and since $z \leq n/2$, we have $s > 0$. Clearly β is a Lyndon word and it has $z-1$ 0s. Now suppose there exists an unstable necklace $\alpha = a_1a_2\cdots a_n$ in $\mathbf{N}_2(z,d)$ such that $f(\alpha) = \beta$. Recall we can write α as $(a_1a_2\cdots a_p)^j a_1a_2\cdots a_i x$ where $x = 1$, $j \geq 1$, and $1 \leq i \leq p$. As noted earlier, if β begins with 001, then so must α. Thus $a_1a_2a_3 = 001$. Consider z', the index of α associated with f. If $z' > 2$, then $f(\alpha)$ must have two subwords of the form 001 and hence is not equal to β. Otherwise $z' = 2$. If $i = z'$ then α ends with 001. In order for $f(\alpha) = \beta$, α must begin with 0011. However this contradicts α being a necklace because the rotation starting with its suffix 001 will be less than α. Thus $z' = 2 < i$. This means that $a_1 = 0$, $a_2 = 0$, and each of $a_3, a_4, \ldots, a_i, x$ is 1. Thus, from the definition of f, $f(\alpha)$ must end 1101 which contradicts β ending with 0101.

$\square$

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