De Bruijn Sequences for the Binary Strings with Maximum Density

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Abstract. A de Bruijn sequence is a circular binary string of length 2^n that contains each binary string of length n exactly once as a substring. A maximum-density de Bruijn sequence is a circular binary string of length $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{m}$ that contains each binary string of length n with density (number of 1s) between 0 and m, inclusively. In this paper we efficiently generate maximum-density de Bruijn sequences for all values of n and m. When $n = 2m + 1$ our result gives a "complement-free de Bruijn sequence" which is a circular binary string of length 2^{n-1} that contains each binary string of length n or its complement exactly once as a substring.

Keywords: de Bruijn sequence, fixed-density de Bruijn sequence, Gray codes, necklaces, Lyndon words, cool-lex order

1 Introduction

Let $\mathbf{B}(n)$ be the set of binary strings of length n. The *density* of a binary string is its number of 1s. Let $\mathbf{B}_d(n)$ be the subset of $\mathbf{B}(n)$ whose strings have density d. Let $\mathbf{B}(n, m) = \mathbf{B}_0(n) \cup \mathbf{B}_1(n) \cup \cdots \cup \mathbf{B}_m(n)$ be subset of $\mathbf{B}(n)$ whose strings have density at most m. A *de Bruijn sequence* (or *de Bruijn cycle*) is a circular binary string of length 2^n that contains each string in $\mathbf{B}(n)$ exactly once as a substring [2]. De Bruijn sequences were introduced by de Bruijn [2] (and earlier [3]) and have many generalizations, variations, and applications. For example, one can refer to the recently published proceedings of the *Generalizations on de Bruijn Sequences and Gray Codes* workshop [6].

In this paper we consider a new generalization of de Bruijn sequences that specifies the maximum density of the substrings. A *maximum-density de Bruijn sequence* is a binary string of length $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{m}$ that contains each string in $\mathbf{B}(n, m)$ exactly once as a substring. For example,

$$0000011000101001$$

is a maximum-density de Bruijn sequence since its 16 substrings of length 5 (including those that "wrap-around") are precisely $\mathbf{B}(5, 3)$. Our main results are (1) an explicit construction of maximum-density de Bruijn sequences for all values of n and m, and (2) an efficient algorithm that generates these de Bruijn sequences.

We make four simple observations involving maximum-density de Bruijn sequences for $\mathbf{B}(n, m)$:

1. A maximum-density de Bruijn sequence is simply a de Bruijn sequence when $n = m$. Constructions and fast algorithms for generating de Bruijn sequences are well-known, so we focus on $m < n$.
2. Reversing the order of the bits in a maximum-density de Bruijn sequence simply gives another maximum-density de Bruijn sequence for the same values of n and m. It is easier to describe our sequences in one order, and then generate them in the reverse order.

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3. Complementing each bit in a maximum-density de Bruijn sequence results in a *minimum-density de Bruijn sequence* for the binary strings of length n whose density is at least $n-m$.
4. A maximum-density de Bruijn sequence is a*complement-free de Bruijn sequence* when $n = 2m+1$. This is because each binary strings of length n either has density at most m or has density at least $n-m$.

Section 2 provides background results. Section 3 describes our construction and proves its correctness. Section 4 provides an algorithm that generates our construction and analyzes its efficiency.

2 Background

A *necklace* is a binary string in its lexicographically smallest rotation. The necklaces over $\mathbf{B}(n)$ and $\mathbf{B}_d(n)$ are denoted $\mathbf{N}(n)$ and $\mathbf{N}_d(n)$, respectively. The *aperiodic prefix* of a string $\alpha = a_1a_2\cdots a_n$ is its shortest prefix $\rho(\alpha) = a_1a_2\cdots a_k$ such that $\rho(\alpha)^{n/k} = \alpha$. As is customary, the previous expression uses exponentiation to refer to repeated concatenation. Observe that if $|\rho(\alpha)| = k$, then α has k distinct rotations. For example, $\rho(0010100101) = 00101$ and 0010100101 is a necklace since it is lexicographically smaller than its other four distinct rotations 0101001010, 1010010100, 0100101001, and 1001010010.

One of the most important results in the study of de Bruijn sequences is due to Fredricksen, Kessler and Maiorana [4, 5] (also see Knuth [7]). These authors proved that a de Bruijn sequence for $\mathbf{B}(n)$ can be constructed by concatenating the aperiodic prefixes of the strings in $\mathbf{N}(n)$ in lexicographic order. For example, the lexicographic order of $\mathbf{N}(6)$ is

$$\begin{aligned}&000000,\ 000001,\ 000011,\ 000101,\ 000111,\ 001001,\ 001011,\\ &\qquad 001101,\ 001111,\ 010101,\ 010111,\ 011011,\ 011111,\ 111111\end{aligned} \tag{1}$$

and so the following is a de Bruijn sequence for $\mathbf{B}(6)$, where $\cdot$ visually separates the aperiodic prefixes from (1) in the concatenation

$$\begin{aligned}&0\cdot 000001\cdot 000011\cdot 000101\cdot 000111\cdot 001001\cdot 001011\cdot\\ &\qquad 001101\cdot 001111\cdot 01\cdot 010111\cdot 011011\cdot 011111\cdot 1.\end{aligned} \tag{2}$$

Although (2) is written linearly, we treat it as a circular string so that its substrings can "wrap-around" from the end to the beginning. Interestingly, (2) is also the lexicographically smallest de Bruijn sequence for $\mathbf{B}(6)$ (when written linearly). Subsequent analysis by Ruskey, Savage, and Wang [8] proved that these lexicographically smallest de Bruijn sequences can be generated efficiently for all values of n.

Recently, it was shown that this *necklace-prefix algorithm* can be modified to create a restricted type of de Bruijn sequence. *Reverse cool-lex order* is a variation of co-lexicographic order that was first defined for $\mathbf{B}_d(n)$ by Ruskey and Williams [11], and has since been generalized to subsets of $\mathbf{B}_d(n)$ including $\mathbf{N}_d(n)$ by Ruskey, Sawada, Williams [10]. In this paper we let $\mathsf{cool}_d(n)$ denote the reverse cool-lex order of $\mathbf{N}_d(n)$. The order for $n=8$ and $d=4$ appears below

$$\begin{aligned}\mathsf{cool}_4(8) = {}&00001111,\ 00011101,\ 00110101,\ 01010101,\ 00101101,\\ &\qquad 00011011,\ 00110011,\ 00101011,\ 00010111,\ 00100111.\end{aligned} \tag{3}$$

Let $\mathsf{dB}_d(n)$ denote the concatenation of the aperiodic prefixes of $\mathsf{cool}_d(n+1)$. For example, the concatenation of the aperiodic prefixes of (3) gives the following

$$\begin{aligned}\mathsf{dB}_4(7) = {}&00001111\cdot 00011101\cdot 00110101\cdot 01\cdot 00101101\cdot\\ &\qquad 00011011\cdot 0011\cdot 00101011\cdot 00010111\cdot 00100111.\end{aligned} \tag{4}$$

Observe that the circular string in (4) contains each string in $\mathbf{B}_3(7) \cup \mathbf{B}_4(7)$ exactly once as a substring, and has no other substrings of length 7. For this reason we describe it as a *dual-density de Bruijn sequence* for $\mathbf{B}_3(7) \cup \mathbf{B}_4(7)$ in this paper[4]. More generally, Ruskey, Sawada, and Williams [9] proved the following result.

Theorem 1. *[9] The circular string $\mathsf{dB}_d(n)$ is a dual-density de Bruijn sequence for $\mathbf{B}_{d-1}(n) \cup \mathbf{B}_d(n)$ when $1 < d < n$.*

In this paper we do not need to completely understand the proof of Theorem 1, but we do need a simple property for its dual-density de Bruijn sequences. In other words, we need to treat each $\mathsf{dB}_d(n)$ as a "gray box". The specific property we need is stated in the following simple lemma involving reverse cool-lex orders that contain at least three necklaces. This lemma follows immediately from equation (5.1) in [9].

Lemma 1. *[9] If $n \geq 5$ and $2 < d < n-1$ then*

- *the first necklace in $\mathsf{cool}_d(n+1)$ is $0^{n-d+1}1^d$, and*
- *the second necklace in $\mathsf{cool}_d(n+1)$ is $0^{n-d}1^{d-1}01$, and*
- *the last necklace in $\mathsf{cool}_d(n+1)$ is $0^x10^y1^{d-1}$*

where $x = \lceil (n+1-d)/2 \rceil$ and $y = \lfloor (n+1-d)/2 \rfloor$. Moreover, each of the necklaces given above are distinct from one another and are aperiodic.

In Section 3 we will be taking apart the dual-density de Bruijn sequence $\mathsf{dB}_d(n)$ around the location of the necklace $0^{n-d+1}1^d$ from $\mathsf{cool}_d(n+1)$. For this reason we make two auxiliary definitions. Let $\mathsf{cool'}_d(n+1)$ equal $\mathsf{cool}_d(n+1)$ except that the first necklace $0^{n-d+1}1^d$ omitted. Similarly, let $\mathsf{dB'}_d(n)$ be the concatenation of the aperiodic prefixes of $\mathsf{cool'}_d(n+1)$. For example, we will be splitting $\mathsf{dB}_4(7)$ in (4) into 00001111 and

$$\mathsf{dB'}_4(7) = 00011101 \cdot 00110101 \cdot 01 \cdot 00101101 \cdot 00011011 \cdot 0011 \cdot 00101011 \cdot 00010111 \cdot 00100111$$

3 Construction

In this section we define a circular string $\mathsf{dB}(n,m)$ of length $1+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{m}$ for $m < n$. Then we prove that $\mathsf{dB}(n,m)$ is a maximum-density de Bruijn sequence for $\mathbf{B}(n,m)$. Table 1 provides examples, and the $m = n$ special case is discussed at the end of the section.

$$\mathsf{dB}(n,m) = \begin{cases} 0\ 0^{n-1}1^2\ 0^{n-3}1^4\ 0^{n-5}1^6\ \cdots\ 0^{n-m+1}1^m\ \mathsf{dB'}_m(n)\ \mathsf{dB'}_{m-2}(n)\ \cdots\ \mathsf{dB'}_2(n) & \text{if } m \text{ is even} \\ \quad 0^n1\ 0^{n-2}1^3\ 0^{n-4}1^5\ \cdots\ 0^{n-m+1}1^m\ \mathsf{dB'}_m(n)\ \mathsf{dB'}_{m-2}(n)\ \cdots\ \mathsf{dB'}_3(n) & \text{if } m \text{ is odd} \end{cases}$$

Theorem 2. *The circular string $\mathsf{dB}(n,m)$ is a maximum-density de Bruijn sequence for $\mathbf{B}(n,m)$ when $m < n$.*

Proof. The result can be easily verified when $n \leq 4$ or $m \leq 2$ (see [9]). We prove the remaining cases by induction on m where $m = 1$ and $m = 2$ provide the base cases. Assume the result holds for $m = k$ and $m = k-1$ and consider $m = k+1 < n$. By induction, $\mathsf{dB}(n,m-2)$ is a maximum-density de Bruijn sequence for $\mathbf{B}(n,m-2)$. The substrings of length n in $\mathsf{dB}(n,m-2)$ and $\mathsf{dB}_m(n)$

[4] The string in (4) is described as a *fixed-density de Bruijn sequence* in [9] since each substring in $\mathbf{B}_3(7) \cup \mathbf{B}_4(7)$ can be uniquely extended to a string in $\mathbf{B}_4(8)$ by appending its 'missing' bit.

Cool-lex orders		Maximum-density de Bruijn sequences				
(even densities)		$\mathsf{dB}(7,0)$	$\mathsf{dB}(7,2)$	$\mathsf{dB}(7,4)$	$\mathsf{dB}(7,6)$	$\mathsf{dB}(7,7)$
	00000000	0	0	0	0	0
	00000011		00000011	00000011	00000011	00000011
	00001111			00001111	00001111	00001111
	00111111				00111111	00111111
	11111111					1
$\mathsf{cool}_0(8)$	$\mathsf{cool'}_0(8)$					
00000000						
$\mathsf{cool}_2(8)$	$\mathsf{cool'}_2(8)$					
00000011						
00000101	00000101		00000101	00000101	00000101	00000101
00001001	00001001		00001001	00001001	00001001	00001001
00010001	00010001		0001	0001	0001	0001
$\mathsf{cool}_4(8)$	$\mathsf{cool'}_4(8)$					
00001111						
00011101	00011101			00011101	00011101	00011101
00110101	00110101			00110101	00110101	00110101
01010101	01010101			01	01	01
00101101	00101101			00101101	00101101	00101101
00011011	00011011			00011011	00011011	00011011
00110011	00110011			0011	0011	0011
00101011	00101011			00101011	00101011	00101011
00010111	00010111			00010111	00010111	00010111
00100111	00100111			00100111	00100111	00100111
$\mathsf{cool}_6(8)$	$\mathsf{cool'}_6(8)$					
00111111						
01110111	01110111				0111	0111
01101111	01101111				01101111	01101111
01011111	01011111				01011111	01011111
$\mathsf{cool}_8(8)$	$\mathsf{cool'}_8(8)$					
11111111						

Table 1. Cool-lex orders of necklaces $\mathsf{cool}_d(8)$ for even d, and how they are used to construct maximum-density de Bruijn sequences $\mathsf{dB}(7,m)$ for $m = 0,2,4,6,7$. For example, $\mathsf{dB}(7,2) =$ 0 00000011 00000101 00001001 0001.

collectively equal $\mathbf{B}(n,m)$, so we can complete the induction by proving that (a) every substring of length n that appears in $\mathsf{dB}(n,m-2)$ appears in $\mathsf{dB}(n,m)$, and (b) every substring of length n that appears in $\mathsf{dB}_m(n)$ appears in $\mathsf{dB}(n,m)$. To do this we consider the difference between $\mathsf{dB}(n,m-2)$ and $\mathsf{dB}(n,m)$ as expressed below using Lemma 1

$$\begin{aligned}\mathsf{dB}(n,m-2) &= \cdots 0^{n-m+3}1^{m-2}\ \mathsf{dB'}_{m-2}(n)\cdots\\ &= \cdots 0^{n-m+3}1^{m-2}\ 0^{n-m+2}1^{m-3}01\cdots \qquad (5)\end{aligned}$$

$$\begin{aligned}\mathsf{dB}(n,m) &= \cdots 0^{n-m+3}1^{m-2}\ 0^{n-m+1}1^{m}\ \mathsf{dB'}_{m}(n)\ \mathsf{dB'}_{m-2}(n)\cdots\\ &= \cdots 0^{n-m+3}1^{m-2}\ \underbrace{0^{n-m+1}1^{m}\ 0^{n-m}1^{m-1}01\ \cdots\ 0^{x}10^{y}1^{m-1}}_{\mathsf{dB}_m(n)}\ 0^{n-m+2}1^{m-3}01\cdots \qquad (6)\end{aligned}$$

where $x = \lceil (n+1-m)/2 \rceil$ and $y = \lfloor (n+1-m)/2 \rfloor$. To prove (a), consider (5) and (6). Observe that the substrings of length n in $0^{n-m+3}1^{m-2}0^{n-m+2}$ from (5) can be found in $0^{n-m+3}1^{m-2}0^{n-m+1}$ from (6), except for $1^{m-2}0^{n-m+2}$. Similarly, the substrings of length n in $1^{m-2}0^{n-m+2}1^{m-3}$ from (5) can be found in $1^{m-1}0^{n-m+2}1^{m-3}$ from (6), and the latter also includes $1^{m-2}0^{n-m+2}$. To prove (b), consider how the insertion of $\mathsf{dB}_m(n)$ into $\mathsf{dB}(n,m)$ in (6) affects its substrings that can no longer "wrap-around" in $\mathsf{dB}_m(n)$. Observe that the substrings of length n in the wrap-around $1^{m-1}0^{n-m+1}$ in $\mathsf{dB}_m(n)$ can all be found in $1^{m-1}0^{n-m+2}$ in (6). Therefore, $\mathsf{dB}(n,m)$ is a maximum-density de Bruijn sequence for $\mathbf{B}(n,m)$. □

As mentioned in Section 1, maximum-density de Bruijn cycles for $\mathsf{dB}(2m+1,m)$ are complement-free de Bruijn cycles for $\mathbf{B}(2m+1)$.

Corollary 1. *The circular string* $\mathsf{dB}(2m+1,m)$ *is a complement-free de Bruijn sequence for* $\mathbf{B}(2m+1)$.

The expression for $\mathsf{dB}(n,m)$ will also produce a de Bruijn sequence when $n=m$ is even. When $n=m$ is odd, an additional 1 (from $\rho(1^{n-1})-1$) needs to be inserted at the appropriate location. These cases are given below for $\mathsf{dB}(n,n)$

$$\mathsf{dB}(n,n) = \begin{cases} 0\ 0^{n-1}1^2\ 0^{n-3}1^4\ 0^{n-5}1^6\ \cdots\ 01^n\quad \mathsf{dB'}_m(n)\ \mathsf{dB'}_{m-2}(n)\ \cdots\ \mathsf{dB'}_2(n) & \text{if } n \text{ is even}\\ 0^n1\ 0^{n-2}1^3\ 0^{n-4}1^5\ \cdots\ 001^{n-1}\ 1\ \mathsf{dB'}_m(n)\ \mathsf{dB'}_{m-1}(n)\ \cdots\ \mathsf{dB'}_3(n) & \text{if } n \text{ is odd.}\end{cases}$$

Our fully-specified construction will now work for all m satisfying $0 \le m \le n$. In other words, our construction of maximum-density de Bruijn sequences fully generalizes the construction of de Bruijn sequences.

Theorem 3. *The circular string* $\mathsf{dB}(n,n)$ *is a de Bruijn sequence for* $\mathbf{B}(n)$.

Proof. **Case one:** $m=n$ is odd. By Theorem 2, $\mathsf{dB}(n,n-1)$ is a maximum-density de Bruijn sequence for $\mathbf{B}(n,n-1)$. By Lemma 1, the difference between $\mathsf{dB}(n,n-1)$ and $\mathsf{dB}(n,n)$ is expressed below

$$\begin{aligned}\mathsf{dB}(n,n-1) &= \cdots 001^{n-1}\ \mathsf{dB'}_{n-2}(n)n-1\cdots \qquad & \mathsf{dB}(n,n) &= \cdots 001^{n-1}\ 1\ \mathsf{dB'}_{n-2}(n)n-1\cdots\\ &= \cdots 001^{n-1}\ 01^{n-2}01\cdots & &= \cdots 001^{n-1}\ 1\ 01^{n-2}01\cdots .\end{aligned}$$

Every substring of length n that appears on the left also appears on the right. The only substring of length n that appears on the right and not the left is 1^n. Therefore, $\mathsf{dB}(n,n)$ is a maximum-density de Bruijn sequence for $\mathbf{B}(n,n)$.

Case two: $m = n$ is even. By Theorem 2, $\mathsf{dB}(n, n-2)$ is a maximum-density de Bruijn sequence for $\mathbf{B}(n, n-2)$. By Lemma 1, the difference between $\mathsf{dB}(n, n-2)$ and $\mathsf{dB}(n, n)$ is expressed below

$$\begin{aligned} \mathsf{dB}(n, n-2) &= \cdots 0001^{n-1}\ \mathsf{dB'}_{n-2}(n) \cdots & \qquad \mathsf{dB}(n, n) &= \cdots 0001^{n-1}\ \mathsf{dB'}_{n-2}(n) \cdots \\ &= \cdots 0001^{n-2}\ 001^{n-3}01 \cdots & &= \cdots 0001^{n-2}\ 01^{n}\ 001^{n-3}01 \cdots . \end{aligned}$$

Every substring of length n that appears on the left also appears on the right. The substrings of length n that appear on the right and not the left are $1^{n-2}01, 1^{n-3}011, \ldots, 01^n, 1^n, 1^{n-1}0$. Therefore, $\mathsf{dB}(n, n)$ is a maximum-density de Bruijn sequence for $\mathbf{B}(n, n)$. □

4 Algorithm

As mentioned earlier, the *reversal* of $\mathsf{dB}(n, m)$, denoted $\mathsf{dB}(n, m)^R$ also yields a maximum-density de Bruijn sequence. To efficiently produce $\mathsf{dB}(n, m)^R$ we can use the recursive cool-lex algorithm described in [12] to produce the reversal of $\mathsf{dB}_d(n)$. In that paper, details are provided to trim each necklace to its longest Lyndon prefix and to output the strings in reverse order. An analysis shows that on average, each n bits can be visited in constant time. There are two data structures used to maintain the current necklace: a string representation $a_1a_2 \cdots a_n$, and a block representation $B_cB_{c-1} \cdots B_1$ where a *block* is defined to be a maximal substring of the form 0^*1^*. A block of the form 0^s1^t is represented by (s, t). Since the number of blocks c is maintained as a global parameter, it is easy to test if the current necklace is of the form 0^*1^*: simply test if $c = 1$. By adding this test, it is a straightforward matter to produce the reversal of $\mathsf{dB'}_d(n)$. To be consistent with the description in [12], the function $\mathsf{Gen}(n-d, d)$ can be used to produce $\mathsf{dB'}_d(n)$. Using this function, the following pseudocode can be used to produce $\mathsf{dB}(n, m)^R$:

```
if $m$ is even then  $start := 2$
else $start := 3$
for $i$ from $start$ by $2$ to $m$ do
    Initialize($n+1, i$)
    Gen($n+1-i, i$)
for $i$ from $m$ by $2$ downto $0$ do Print( $1^i0^{n+i-1}$ )
if $m$ is even then  Print( $0$ )
```

The $\mathsf{Initialize}(n+1, i)$ function sets $a_1a_2 \cdots a_{n+1}$ to $0^{n+1-i}1^i$, sets $c = 1$ and sets $B_1 = (n+1-i, i)$. The first time it is called it requires $O(n)$ time, and for each subsequent call, the updates can be performed in $O(1)$ time. Also note that the string visited by the $\mathsf{Print}()$ function can also be updated in constant time after the first string is visited. Since the extra work outside the calls to Gen requires $O(n)$ time, and because the number of bits in $\mathsf{dB}(n, m)$ is $\Omega(n^2)$ where $1 < m < n$, we obtain the following theorem.

Theorem 4. *The maximum-density de Bruijn sequence* $\mathsf{dB}(n, m)^R$ *can be generated in constant amortized time for each* n *bits visited, where* $1 < m < n$.

5 Open Problems

A natural open problem is to efficiently construct *density-range de Bruijn sequences* for the binary strings of length n whose density is between i and j (inclusively) for any $0 \le i < j \le n$.

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