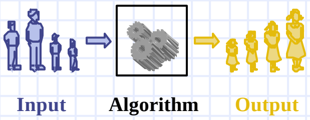
# Analysis of Algorithms



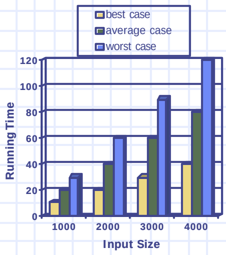

## Outline and Reading

- Running time (§3.1)
- Pseudo-code (§3.2)
- Counting primitive operations (§3.3-3.5)
- Asymptotic notation (§3.6)
- Asymptotic analysis (§3.7)
- Case study



## Running Time

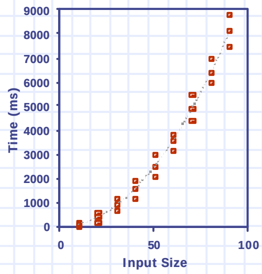
- The running time of an algorithm varies with the input and typically grows with the input size
- Average case difficult to determine
- We focus on the worst case running time
  - Easier to analyze
  - Crucial to applications such as games, finance and robotics






## Experimental Studies

- Write a program implementing the algorithm
- Run the program with inputs of varying size and composition
- Use a method like System.currentTimeMillis() to get an accurate measure of the actual running time
- Plot the results






## Limitations of Experiments

- It is necessary to implement the algorithm, which may be difficult
- Results may not be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the same hardware and software environments must be used



## Theoretical Analysis

- Uses a high-level description of the algorithm instead of an implementation
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment

# Pseudocode

- High-level description of an algorithm
- More structured than English prose
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues

Example: find max element of an array

**Algorithm** *arrayMax*(*A*, *n*)
  **Input** array *A* of *n* integers
  **Output** maximum element of *A*

  *currentMax* ← *A*[0]
  **for** *i* ← 1 **to** *n* − 1 **do**
    **if** *A*[*i*] > *currentMax* **then**
      *currentMax* ← *A*[*i*]
  **return** *currentMax*



# Pseudocode Details

- Control flow
  - **if** … **then** … [**else** …]
  - **while** … **do** …
  - **repeat** … **until** …
  - **for** … **do** …
  - Indentation replaces braces
- Method declaration
  **Algorithm** *method* (*arg* [, *arg*…])
  **Input** …
  **Output** …
- Method call
  *var.method* (*arg* [, *arg*…])
- Return value
  **return** *expression*
- Expressions
  ← Assignment (like = in Java)
  = Equality testing (like == in Java)
  $n^2$ Superscripts and other mathematical formatting allowed



# Primitive Operations

- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely independent from the programming language
- Exact definition not important (we will see why later)
- Examples:
  - Evaluating an expression
  - Assigning a value to a variable
  - Indexing into an array
  - Calling a method
  - Returning from a method



# Counting Primitive Operations

- By inspecting the pseudocode, we can determine the maximum number of primitive operations executed by an algorithm, as a function of the input size

| **Algorithm** *arrayMax*(*A*, *n*) | # operations |
|---|---|
| *currentMax* ← *A*[0] | 2 |
| **for** *i* ← 1 **to** *n* − 1 **do** | $2 + n$ |
| **if** *A*[*i*] > *currentMax* **then** | $2(n - 1)$ |
| *currentMax* ← *A*[*i*] | $2(n - 1)$ |
| { increment counter *i* } | $2(n - 1)$ |
| **return** *currentMax* | 1 |
| Total | $7n - 1$ |



# Estimating Running Time

- Algorithm ***arrayMax*** executes $7n - 1$ primitive operations in the worst case
- Define
  - $a$ Time taken by the fastest primitive operation
  - $b$ Time taken by the slowest primitive operation
- Let $T(n)$ be the actual worst-case running time of ***arrayMax***. We have
  $$a(7n - 1) \le T(n) \le b(7n - 1)$$
- Hence, the running time $T(n)$ is bounded by two linear functions

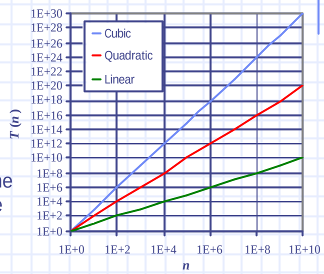


# Growth Rate of Running Time

- Changing the hardware/ software environment
  - Affects $T(n)$ by a constant factor, but
  - Does not alter the growth rate of $T(n)$
- The linear growth rate of the running time $T(n)$ is an intrinsic property of algorithm ***arrayMax***

## Growth Rates

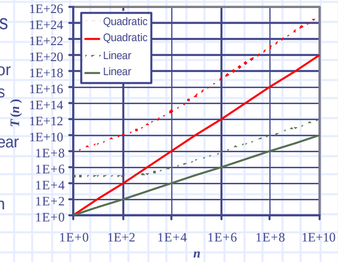
- Growth rates of functions:
  - Linear $\approx n$
  - Quadratic $\approx n^2$
  - Cubic $\approx n^3$
- In a log-log chart, the slope of the line corresponds to the growth rate of the function






## Constant Factors

- The growth rate is not affected by
  - constant factors or
  - lower-order terms
- Examples
  - $10^2n + 10^5$ is a linear function
  - $10^5n^2 + 10^8n$ is a quadratic function

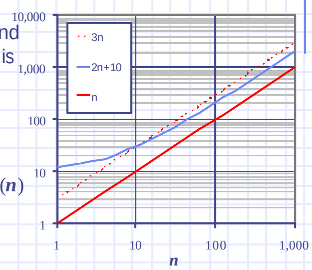





## Big-Oh Notation

- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants $c$ and $n_0$ such that

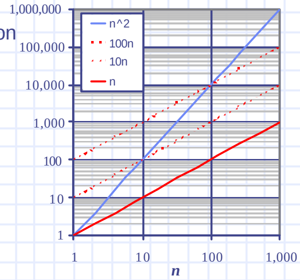
  $f(n) \leq cg(n)$ for $n \geq n_0$
- Example: $2n + 10$ is $O(n)$
  - $2n + 10 \leq cn$
  - $(c - 2)\, n \geq 10$
  - $n \geq 10/(c - 2)$
  - Pick $c = 3$ and $n_0 = 10$






## Big-Oh Notation (cont.)

- Example: the function $n^2$ is not $O(n)$
  - $n^2 \leq cn$
  - $n \leq c$
  - The above inequality cannot be satisfied since $c$ must be a constant






## Big-Oh and Growth Rate

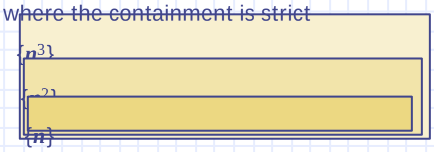
- The big-Oh notation gives an upper bound on the growth rate of a function
- The statement "$f(n)$ is $O(g(n))$" means that the growth rate of $f(n)$ is no more than the growth rate of $g(n)$
- We can use the big-Oh notation to rank functions according to their growth rate

| | $f(n)$ is $O(g(n))$ | $g(n)$ is $O(f(n))$ |
|---|---|---|
| $g(n)$ grows more | Yes | No |
| $f(n)$ grows more | No | Yes |
| Same growth | Yes | Yes |



## Classes of Functions

- Let $\{g(n)\}$ denote the class (set) of functions that are $O(g(n))$
- We have $\{n\} \subset \{n^2\} \subset \{n^3\} \subset \{n^4\} \subset \{n^5\} \subset \ldots$ where the containment is strict



## Big-Oh Rules

- If is $f(n)$ a polynomial of degree $d$, then $f(n)$ is $O(n^d)$, i.e.,
  1. Drop lower-order terms
  2. Drop constant factors
- Use the smallest possible class of functions
  - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"
- Use the simplest expression of the class
  - Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"



## Asymptotic Algorithm Analysis

- The asymptotic analysis of an algorithm determines the running time in big -Oh notation
- To perform the asymptotic analysis
  - We find the worst-case number of primitive operations executed as a function of the input size
  - We express this function with big-Oh notation
- Example:
  - We determine that algorithm ***arrayMax*** executes at most $7n - 1$ primitive operations
  - We say that algorithm ***arrayMax*** "runs in $O(n)$ time"
- Since constant factors and lower-order terms are eventually dropped anyhow, we can disregard them when counting primitive operations

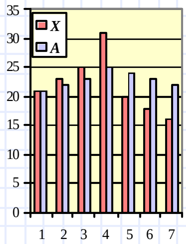


## Computing Prefix Averages

- We further illustrate asymptotic analysis with two algorithms for prefix averages
- The $i$-th prefix average of an array $X$ is average of the first $(i + 1)$ elements of $X$

$A[i] = (X[0] + X[1] + \ldots + X[i])/(i+1)$

- Computing the array $A$ of prefix averages of another array $X$ has applications to financial analysis






## Prefix Averages (Quadratic)

- The following algorithm computes prefix averages in quadratic time by applying the definition

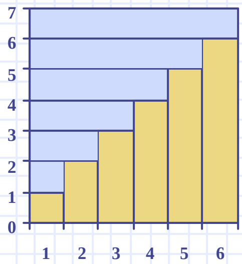
**Algorithm** ***prefixAverages1(X, n)***
**Input** array $X$ of $n$ integers
**Output** array $A$ of prefix averages of $X$

| | #operations |
|---|---|
| $A \leftarrow$ new array of $n$ integers | $n$ |
| **for** $i \leftarrow 0$ **to** $n - 1$ **do** | $n$ |
| $\quad s \leftarrow X[0]$ | $n$ |
| $\quad$ **for** $j \leftarrow 1$ **to** $i$ **do** | $1 + 2 + \ldots + (n - 1)$ |
| $\quad\quad s \leftarrow s + X[j]$ | $1 + 2 + \ldots + (n - 1)$ |
| $\quad A[i] \leftarrow s / (i + 1)$ | $n$ |
| **return** $A$ | 1 |



## Arithmetic Progression

- The running time of ***prefixAverages1*** is $O(1 + 2 + \ldots + n)$
- The sum of the first $n$ integers is $n(n + 1) / 2$
  - There is a simple visual proof of this fact
- Thus, algorithm ***prefixAverages1*** runs in $O(n^2)$ time






## Prefix Averages (Linear)

- The following algorithm computes prefix averages in linear time by keeping a running sum

**Algorithm** ***prefixAverages2(X, n)***
**Input** array $X$ of $n$ integers
**Output** array $A$ of prefix averages of $X$

| | #operations |
|---|---|
| $A \leftarrow$ new array of $n$ integers | $n$ |
| $s \leftarrow 0$ | 1 |
| **for** $i \leftarrow 0$ **to** $n - 1$ **do** | $n$ |
| $\quad s \leftarrow s + X[i]$ | $n$ |
| $\quad A[i] \leftarrow s / (i + 1)$ | $n$ |
| **return** $A$ | 1 |

- Algorithm ***prefixAverages2*** runs in $O(n)$ time

# Asymptotic Notation *(terminology):*

- Special classes of algorithms:
  - ***logarithmic***: **O(log n)**
  - ***linear***: **O(n)**
  - ***quadratic***: **O($n^2$)**
  - ***polynomial***: **O($n^k$), k = 1**
  - ***exponential***: **O($a^n$), n > 1**
- "Relatives" of the Big-Oh
  - Ω (f(n)): Big Omega--asymptotic *lower* bound
  - Θ (f(n)): Big Theta--asymptotic *tight* bound



A table of functions wrt input n, assume that each primitive operation takes one microsecond (1 second = $10^6$ microsecond).

| O(g(n)) | 1 Second | 1 Hour | 1 Month | 1 Century |
|---|---|---|---|---|
| $\log_2 n$ | $\approx 10^{300000}$ | $\approx 10^{10^9}$ | $\approx 10^{0.8*10^{12}}$ | $\approx 10^{10^{15}}$ |
| *Ön* | $\approx 10^{12}$ | $\approx 1.3*10^{19}$ | $\approx 6.8*10^{24}$ | $\approx 9.7*10^{30}$ |
| *n* | $10^6$ | $3.6*10^9$ | $\approx 2.6*10^{12}$ | $\approx 3.12*10^{15}$ |
| $n \log_2 n$ | $\approx 10^5$ | $\approx 10^9$ | $\approx 10^{11}$ | $\approx 10^{14}$ |
| $n^2$ | 1000 | $6*10^4$ | $\approx 1.6*10^6$ | $\approx 5.6*10^7$ |
| $n^3$ | 100 | ≈ 1500 | ≈ 14000 | ≈ 1500000 |
| $2^n$ | 19 | 31 | 41 | 51 |
| *n!* | 9 | 12 | 15 | 17 |