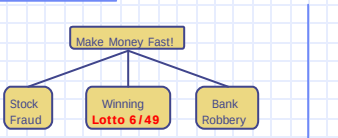
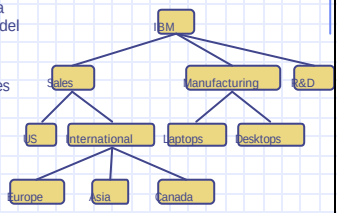
Trees



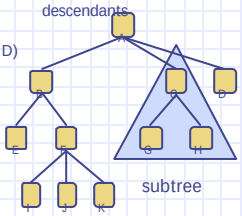
What is a Tree

- In computer science, a tree is an abstract model of a hierarchical structure
- A tree consists of nodes with a parent-child relation
- Applications:
  - Organization charts
  - File systems
  - Programming environments



Tree Terminology

- Root: node without parent (A)
- Internal node: node with at least one child (A, B, C, F)
- External node (a.k.a. leaf): node without children (E, I, J, K, G, H, D)
- Ancestors of a node: parent, grandparent, grand-grandparent, etc.
- Depth of a node: number of ancestors
- Height of a tree: maximum depth of any node (3)
- Descendant of a node: child, grandchild, grand-grandchild, etc.
- Subtree: tree consisting of a node and its descendants



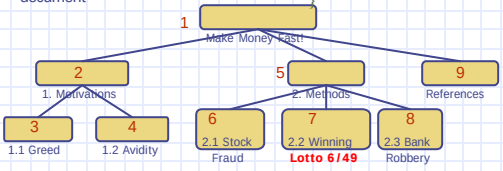
Tree ADT

- We use positions to abstract nodes
- Generic functions:
  - int size()
  - boolean isEmpty()
- Access functions:
  - node root(tree)
  - node parent(p)
  - node left_child(p)
  - node right_child(p)
- Query functions:
  - boolean isInternal(p)
  - boolean isExternal(p)
  - boolean isRoot(p)
  - int height(tree)
- Additional update functions may be defined by data structures implementing the Tree ADT

Preorder Traversal

- A traversal visits the nodes of a tree in a systematic manner
- In a preorder traversal, a node is visited before its descendants
- Application: print a structured document

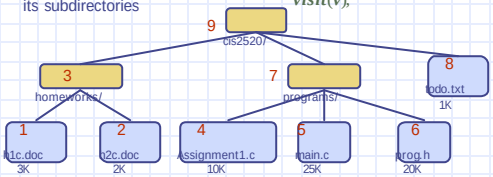
```
function preOrder(v)
    visit(v);
    for (each child w of v){
        preorder (w);
    }
```



Postorder Traversal

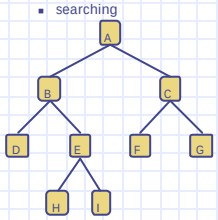
- In a postorder traversal, a node is visited after its descendants
- Application: compute space used by files in a directory and its subdirectories

```
function postOrder(v)
    for (each child w of v){
        postOrder (w);
    }
    visit(v);
```



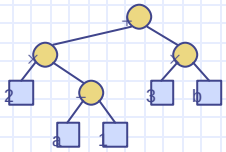
Binary Tree

- A binary tree is a tree with the following properties:
  - Each internal node has two children
  - The children of a node are an ordered pair
- We call the children of an internal node left child and right child
- Alternative recursive definition: a binary tree is either
  - a tree consisting of a single node, or
  - a tree whose root has an ordered pair of children, each of which is a binary tree
- Applications:
  - arithmetic expressions
  - decision processes
  - searching



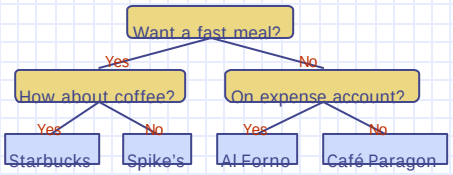
Arithmetic Expression Tree

- Binary tree associated with an arithmetic expression
  - internal nodes: operators
  - external nodes: operands
- Example: arithmetic expression tree for the expression $(2 \times (a - 1) + (3 \times b))$



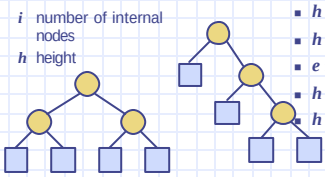
Decision Tree

- Binary tree associated with a decision process
  - internal nodes: questions with yes/no answer
  - external nodes: decisions
- Example: dining decision



Properties of Binary Trees

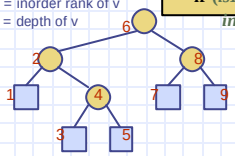
- Notation
  - n number of nodes
  - e number of external nodes
  - i number of internal nodes
  - h height
- Properties:
  - $e = i + 1$
  - $n = 2e - 1$
  - $h \leq i$
  - $h \leq (n - 1)/2$
  - $e \leq 2^h$
  - $h \geq \log_2 e$
  - $h \geq \log_2 (n + 1) - 1$



Inorder Traversal

- In an inorder traversal a node is visited after its left subtree and before its right subtree
- Application: draw a binary tree
  - x(v) = inorder rank of v
  - y(v) = depth of v

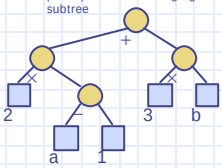
```
function inOrder(v)
    if (isInternal (v)){
        inOrder (leftChild (v));}
    visit(v);
    if (isInternal (v)){
        inOrder (rightChild (v));}
```



Print Arithmetic Expressions

- Specialization of an inorder traversal
  - print "(" before traversing left subtree
  - print operand or operator when visiting node
  - print ")" after traversing right subtree

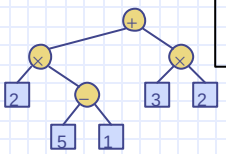
```
function printExpression(v)
    if (isInternal (v)){
        printf("%c", `(');
        inOrder (leftChild (v));
    }
    print(v); // depends on v's data
    if (isInternal (v)){
        inOrder (rightChild (v));
        printf("%c", `)');
    }
```



$((2 \times (a - 1)) + (3 \times b))$

Evaluate Arithmetic Expressions

- Specialization of a postorder traversal
  - recursive method returning the value of a subtree
  - when visiting an internal node, combine the values of the subtrees




```
function evalExpr(v)
    if (isExternal (v)){
        return v.value ();
    }
    else{
        x = evalExpr(leftChild (v));
        y = evalExpr(rightChild (v));
        à ← operator stored at v;
        return x à y;
    }
```


Creativity: pathLength(tree) = Σ depth(v) ∀v ∈ tree

```
function pathLength(v, n)
Input: a tree node v and an initial value n
Output: the pathLength of the tree with root v
Usage: pl = pathLength(root, 0);
    if (isExternal (v)){ return n; }
    else{
        return (n +
          pathLength(leftChild (v), n + 1) +
          pathLength(rightChild (v), n + 1) );
    }
```


Priority Queue ADT

- A priority queue stores a collection of items
- An item is a pair (key, element)
- Main methods of the Priority Queue ADT
  - insertItem(k, o) inserts an item with key k and element o
  - removeMin() removes the item with smallest key and returns its element
- Additional methods
  - minKey() returns, but does not remove, the smallest key of an item
  - minElement() returns, but does not remove, the element of an item with smallest key
  - size(), isEmpty()
- Applications:
  - Standby flyers
  - Auctions
  - Stock market

Example: Priority Queue

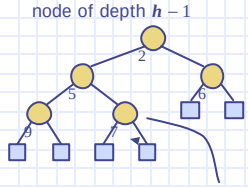
Operator	Output	Priority Queue
insertItem(5, A)	–	(5,A)
insertItem(9, C)	–	(5,A),(9,C)
insertItem(3, B)	–	(3,B),(5,A),(9,C)
insertItem(7, D)	–	(3,B),(5,A),(7,D),(9,C)
minElement()	B	(3,B),(5,A),(7,D),(9,C)
minKey()	3	(3,B),(5,A),(7,D),(9,C)
removeMin()	B	(5,A),(7,D),(9,C)
size()	3	(5,A),(7,D),(9,C)
removeMin()	A	(7,D),(9,C)
removeMin()	D	(9,C)
removeMin()	C	
removeMin()	error	
isEmpty()	true	

Total Order Relation

- Keys in a priority queue can be arbitrary objects on which an order is defined
- Two distinct items in a priority queue can have the same key
- Mathematical concept of total order relation ≤
  - Reflexive property: $x \leq x$
  - Antisymmetric property: $x \leq y \wedge y \leq x \Rightarrow x = y$
  - Transitive property: $x \leq y \wedge y \leq z \Rightarrow x \leq z$

What is a heap

- A heap is a binary tree storing keys at its internal nodes and satisfying the following properties:
  - Heap-Order: for every internal node v other than the root, $key(v) \geq key(parent(v))$
  - Complete Binary Tree: let h be the height of the heap
    - for $i = 0, \ldots, h-1$, there are 2^i nodes of depth i
    - at depth $h-1$, the internal nodes are to the left of the external nodes
- The last node of a heap is the rightmost internal node of depth $h-1$

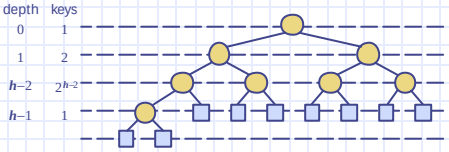


last node

Height of a Heap

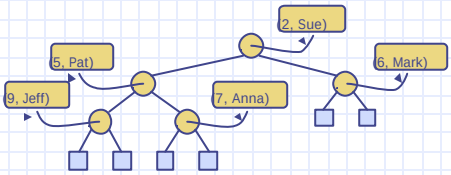
- Theorem: A heap storing n keys has height $O(\log n)$

  Proof: (we apply the complete binary tree property)
  - Let h be the height of a heap storing n keys
  - Since there are 2^i keys at depth $i = 0, \ldots, h-2$ and at least one key at depth $h-1$, we have $n \geq 1 + 2 + 4 + \ldots + 2^{h-2} + 1$
  - Thus, $n \geq 2^{h-1}$, i.e., $h \leq \log n + 1$



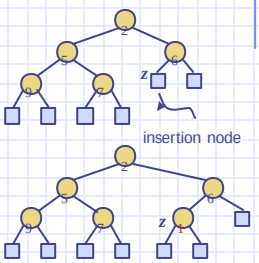
Heaps and Priority Queues

- We can use a heap to implement a priority queue
- We store a (key, element) item at each internal node
- We keep track of the position of the last node
- For simplicity, we show only the keys in the pictures



Insertion into a Heap

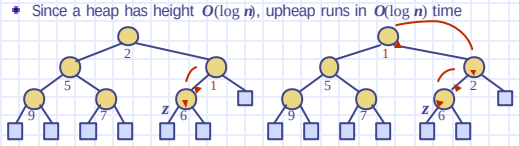
- The insertion algorithm consists of three steps
  - Find the insertion position z (the new last node)
  - Store k at z and expand z into an internal node
  - Restore the heap-order property (discussed next)




```
void insert(Heap *H, ItemType ItemToInsert) {
NodeType *N;                    /* Pointer to the new node to be inserted to H */
NodeType *P;                    /* Let P be the pointer to the new node */
    N = (create a new node with ItemToInsert);
    if (*H is not empty) {
        /* Find the position to hold the new node */
        P = (the pointer to the new node);
        (P's value) = N;
        /* Reheapify the values in the remaining nodes of H starting at the root, R */
        if (H is not empty) {
            (Reheapify the heap H starting at node R);
        }
    }
    else {
        (Root in H) = N;
    }
    return;
}
```

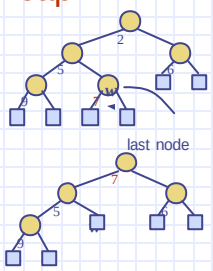

Upheap

- After the insertion of a new key k, the heap-order property may be violated
- Algorithm upheap restores the heap-order property by swapping k along an upward path from the insertion node
- Upheap terminates when the key k reaches the root or a node whose parent has a key smaller than or equal to k
- Since a heap has height $O(\log n)$, upheap runs in $O(\log n)$ time



Removal from a Heap

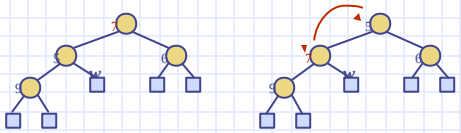
- The removal algorithm consists of three steps
  - Replace the root key with the key of the last node w
  - Delete w
  - Restore the heap-order property (discussed next)



```
ItemType remove(Heap *H) {
NodeType  L;                    /* let L be the last node of H in level order */
NodeType  R;                    /* R is used refer to the root node of H */
ItemType  ItemToRemove;         /* temporarily stores item to remove */
    if (*H is not empty) {
        /* Remove the highest priority item which is stored in H's root node, R */
        ItemToRemove = (the value stored in the root node, R, of H);
        /* Move L's value into the root of H, and delete L */
        (R's value) = (the value in last node L);
        (delete node L);
        /* Reheapify the values in the remaining nodes of H starting at the root, R */
        if (H is not empty) {
          (Reheapify the heap H starting at node R);
        }
|       return (ItemToRemove);
    }
}
```

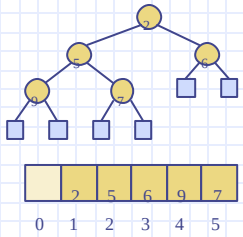

Downheap

- After replacing the root key with the key k of the last node, the heap-order property may be violated
- Algorithm downheap restores the heap-order property by swapping key k along a downward path from the root
- Downheap terminates when key k reaches a leaf or a node whose children have keys greater than or equal to k
- Since a heap has height $O(\log n)$, downheap runs in $O(\log n)$ time



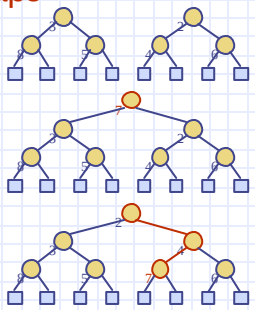
Array-based Implementation

- We can represent a heap with n keys by means of an array of length $n+1$
- For the node at rank i
  - the left child is at rank $2i$
  - the right child is at rank $2i+1$
- Links between nodes are not explicitly stored
- The leaves are not represented
- The cell of at rank 0 is not used
- Operation insert corresponds to inserting at rank $n+1$
- Operation remove corresponds to removing at rank 1



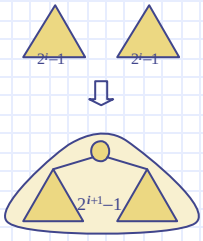
Merging Two Heaps

- We are given two two heaps and a key k
- We create a new heap with the root node storing k and with the two heaps as subtrees
- We perform downheap to restore the heap-order property



Bottom-up Heap Construction

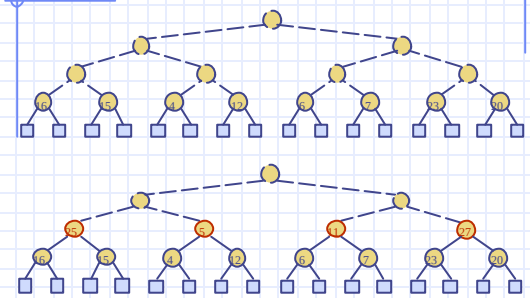
- We can construct a heap storing n given keys in using a bottom-up construction with $\log n$ phases
- In phase i, pairs of heaps with 2^i-1 keys are merged into heaps with $2^{i+1}-1$ keys



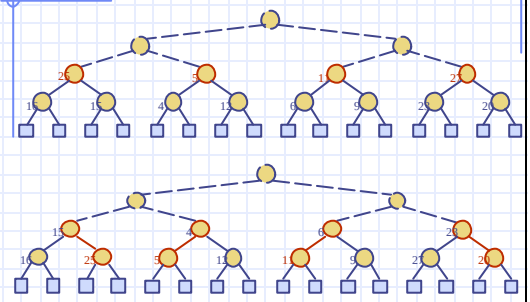
Bottom-up Algorithm:

```
function BottomUpHeap(S, n):
 Input: An array S storing n = 2^h – 1 keys
 Output: A heap T storing the keys in S
if (n == 0) then return NULL;
k = S[0];
S1 = S[1 .. (n-1)/2];       /* split S into two sub-arrays */
S2 = S[(n-1)/2+1 .. n];
T1 = BottomUpHeap(S1);
T2 = BottomUpHeap(S2);
T = treeNode(k, T1, T2);
DownHeap(T);
return T;
```

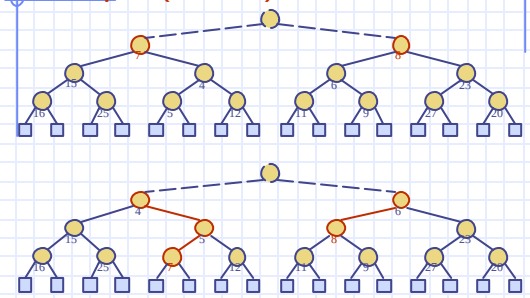
Example



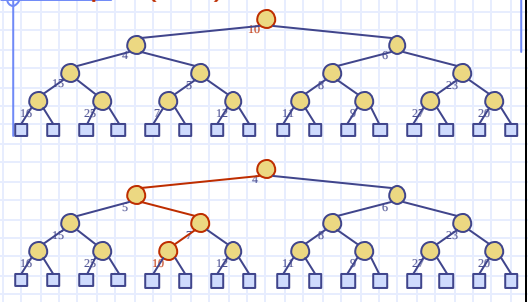
Example (contd.)



Example (contd.)

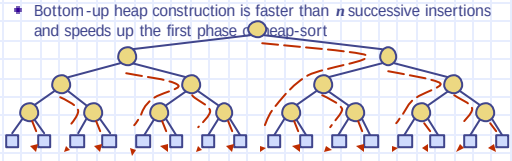


Example (end)



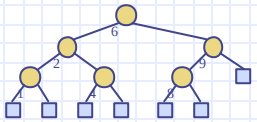
Analysis

- We visualize the worst-case time of a downheap with a proxy path that goes first right and then repeatedly goes left until the bottom of the heap (this path may differ from the actual downheap path)
- Since each node is traversed by at most two proxy paths, the total number of nodes of the proxy paths is $O(n)$
- Thus, bottom-up heap construction runs in $O(n)$ time
- Bottom-up heap construction is faster than n successive insertions and speeds up the first phase of heap-sort



Binary Search Tree

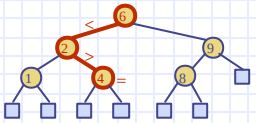
- A binary search tree is a binary tree storing keys (or key-element pairs) at its internal nodes and satisfying the following property:
  - Let v be a tree node, and L, R be subtrees such that L is the left subtree of v and R is the right subtree of v. We have $keys(L) \leq key(v) \leq keys(R)$
- External nodes do not store items (NULL's)
- An inorder traversal of a binary search trees visits the keys in increasing order



Search

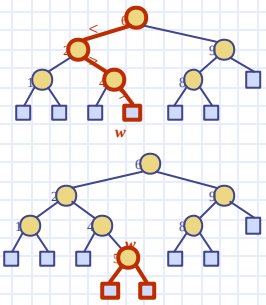
- To search for a key k, we trace a downward path starting at the root
- The next node visited depends on the outcome of the comparison of k with the key of the current node
- If we reach a leaf, the key is not found and we return NO_SUCH_KEY
- Example: findElement(4, root)

```
function findElement(k, v)
    if (isExternal (v)){
        return NO_SUCH_KEY; }
    if (k < key(v)) {
        return findElement(k, leftChild(v));}
    else if (k = key(v)){
        return element(v); }
    else { // k > key(v)
        return findElement(k, rightChild (v)); }
```



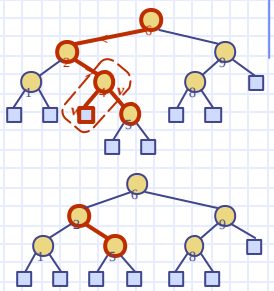
Insertion

- To perform operation insertItem (k, o), we search for key k
- Assume k is not already in the tree, and let w be the leaf reached by the search
- We insert k at node w and expand w into an internal node
- Example: insert 5



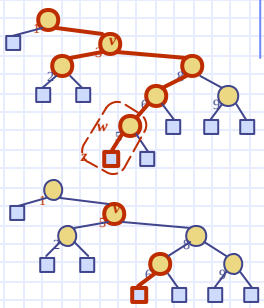
Deletion

- To perform operation removeElement (k), we search for key k
- Assume key k is in the tree, and let v be the node storing k
- If node v has a leaf child w *(a NULL subtree)*, we remove v and w from the tree with operation removeAboveExternal(w)
- Example: remove 4



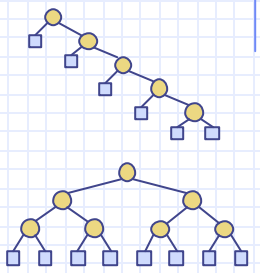
Deletion (cont.)

- We consider the case where the key k to be removed is stored at a node v whose children are both internal
  - we find the internal node w that follows v in an inorder traversal
  - we copy $key(w)$ into node v
  - we remove node w and its left child z (which must be a leaf) by means of operation removeAboveExternal (z)
- Example: remove 3



Performance

- Consider a dictionary with n items implemented by means of a binary search tree of height h
  - the space used is $O(n)$
  - methods findElement , insertItem and removeElement take $O(h)$ time
- The height h is $O(n)$ in the worst case and $O(\log n)$ in the best case



Trees - Implementation

- Data structure

```
typedef struct node{
      void *item;
      struct node *left;
      struct node *right;
} node;


Typedef struct {
      node* root;
      ……
} tree;
```

Trees - Implementation

```
extern int keyCmp( void *a, void *b );
/* Returns -1, 0, 1 for a < b, a == b, a > b */

void *find( node* np, void *key ) {
   if ( np == NULL) return NULL;
   switch( keyCmp( key, np->item) ) {
      case -1 : return find( np->left, key );
      case 0:   return np->item;
      case +1 : return find( np->right, key );
      }
   }

void *findInTree( tree t, void *key ) {
   return find( t.root, key );
   }
```

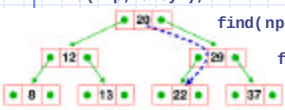
Less, search left

Greater, search right

Trees - Implementation

- Example:
  - key = 22;
    if (findInTree(t.root , &key)) ….

```
find( np, &key );
```




```
find(np->right,&key );
find(np->left,&key );
return np->item;
```


Trees - Addition

- Add 21 to the tree

- We need at most $h+1$ comparisons
- Create a new node (constant time)
- So addition to a tree takes time proportional to $\log n$

implementation

```
void insert( node **t, node *new ) {
  node base = *t;
  if ( base == NULL ) {
     *t = new; return; }
  else {
     if( keyLess(new->item, base->item) )
        insert( &(base->left), new );
     else
        insert( &(base->right), new );
  }
}

void addToTree( tree t, void *item ) {
        node* new;
        new = (node*) malloc(sizeof(struct t_node));
        new->item = item;
        new->left = new->right = NULL;
        insert( &(t.root), new );
}
```